MUZAFFARPUR INSTITUTE OF TECHNOLOGY, MUZAFFARPUR, BIHAR – 842003

(Under the department of Science & Technology, Bihar, Patna)

B.Tech 1th Semester Mid-Term Examination, 2018

Mathematics-I

Time: 2 hours **Branch - CE+ME+IT+LT** **Full Marks: 20**

Attempt any four questions out of which Question No. 1 is compulsory.

1. Chose the correct option of the following: (any five)

(a) If the series $\sum u_n$ is convergent, then $\lim_{n\to\infty} u_n$ is

(i) 0 (ii) ∞ (iii) 1 (iv) -1

(b) Series $\sum\left(\frac{1}{n^{3/2}}\right)$ is

(i) Divergent (ii) Not bounded below (iii) Convergent (iv) None of these

(c) The improper integral $\int_1^\infty \frac{dx}{x^p}$ converges for

(i) $p < 1$ (ii) $p > 1$ (iii) $p = 1$ (iv) All of the above

(d) $\Gamma(-3.5) =$

(i) $16\sqrt{\pi}$ (ii) $8\sqrt{\pi}$ (iii) $\frac{16\sqrt{\pi}}{105}$ (iv) $\Gamma(n)$ not defined for $n < 0$

(e) If a Square matrix A has an eigen value λ, then eigen value of matrix $(KA)^T$, where $K \neq 0$ is

(i) $\frac{\lambda}{K}$ (ii) $\frac{K}{\lambda}$ (iii) $K\lambda$ (iv) None of these

(f) If A is a skew-symmetric matrix of odd order then the $|A|$ is

(i) -1 (ii) 0 (iii) 1 (iv) None of these.

(g) The Rank of the matrix $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is equal to

(i) 0 (ii) 1 (iii) 2 (iv) 3

(a)
2. Show that the series $1+r+r^2+r^3+\cdots\ (r>0)$ converges if $r<1$ and diverges if $r\geq 1$.

Sol^n:— $S_n = 1+r+r^2+\cdots+r^{n-1}$ —— (1)

$$S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

Case I:— Suppose $r<1$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r} - \frac{1}{1-r}\lim_{n\to\infty} r^n$$

$$= \frac{1}{1-r} \qquad \left[\because \lim_{n\to\infty} r^n = 0 \text{ if } |r|<1\right]$$

Thus $\langle S_n\rangle$ is a convergent seq^n. so the given series is convergent.

Case II:— Suppose $r=1$

Then $S_n = 1+1+1+\cdots+1 = n$ by (1)

clearly, the seq^n $\langle S_n\rangle = \langle n\rangle = \langle 1,2,3,\ldots\rangle$ is divergent.

So, the given series is divergent.

Case III:—

Suppose $r>1$

Then $S_n > n$ by (1)

Thus $\lim_{n\to\infty} S_n = +\infty$

So, the series is divergent.

2 (b)

Q:– Test for convergence of the series

$$\sum \left(\sqrt{n^3+1} - \sqrt{n^3}\right).$$

Sol^n:–

$$u_n = \sqrt{n^3+1} - \sqrt{n^3}$$

$$= \frac{\left(\sqrt{n^3+1} - \sqrt{n^3}\right)\left(\sqrt{n^3+1} + \sqrt{n^3}\right)}{\sqrt{n^3+1} - \sqrt{n^3}}$$

$$= \frac{(\cancel{n^3}+1) - \cancel{n^3}}{\sqrt{n^3+1} - \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1} - \sqrt{n^3}}$$

Let $v_n = \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$

Now,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+1} - \sqrt{n^3}}$$

$$= \frac{1}{2} \neq 0 \text{ and finite}$$

So, By limit form test, $\sum u_n$ and $\sum v_n$ converges or diverges together.

Since $\sum v_n = \sum \frac{1}{n^{3/2}}$ is convergent by p-series, where $p>1$

So the given series $\sum u_n$ is also convergent.

③

Ques :-

Evaluate $\int_0^\infty e^{-ax} \cdot x^{m-1} \sin bx \, dx$ in term of Gamma function.

Solution

We have : $\Gamma(m) = \int_0^\infty e^{-x} \cdot x^{m-1} \, dx$.

Put $x = at$

$dx = a\,dt$

As $x \to 0 \Rightarrow t \to 0$

and $x \to \infty \Rightarrow t \to \infty$.

$$\therefore \Gamma(m) = \int_0^\infty e^{-at} \cdot (at)^{m-1} \cdot a\,dt$$

$$= a^m \int_0^\infty e^{-at} \, t^{m-1} \, dt.$$

$$\therefore \int_0^\infty e^{-at} \cdot t^{m-1} \, dt = \frac{\Gamma(m)}{a^m} \quad \text{——— (1)}$$

Now $\because I = \int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx$

$$= \int_0^\infty e^{-ax} \cdot x^{m-1} \cdot (\text{Imaginary Part of } e^{ibx}) \, dx$$

$$= I.P \text{ of } \int_0^\infty e^{-(a-ib)x} \cdot x^{m-1} \, dx.$$

$$= I.P \text{ of } \frac{\Gamma(m)}{(a-ib)^m} \quad [\text{using (1)}].$$

$$= I.P \text{ of } \frac{\Gamma(m)}{r^m (\cos\theta - i\sin\theta)^m}.$$

$$= I.P \text{ of } \frac{\Gamma(m)}{r^m (\cos m\theta - i\sin m\theta)}$$

$$= I.P \text{ of } \frac{\Gamma(m)}{r^m (\cos m\theta - i\sin m\theta)} \times \frac{(\cos m\theta + i\sin m\theta)}{(\cos m\theta + i\sin m\theta)}.$$

$$= I.P \text{ of } \frac{\Gamma(m) \cdot (\cos m\theta + i\sin m\theta)}{r^m (\cos^2 m\theta + \sin^2 m\theta)}.$$

$$= I.P \text{ of } \frac{\Gamma(m) \cdot (\cos m\theta + i\sin m\theta)}{r^m}.$$

$$= \frac{\Gamma(m) \cdot \sin m\theta}{r^m}$$

(4) Ques: Evaluate the following improper integrals

(ii) $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx$ (i) $\int_0^\infty e^{-x^3}\, dx.$

in terms of Gamma functions.

Soln. (ii) $\int_0^\infty \sqrt{x}.\, e^{-x^2}\, dx.$

put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{1}{2\sqrt{t}}\, dt.$

As $x \to 0 \Rightarrow t \to 0$ and $x \to \infty \Rightarrow t \to \infty$

$\therefore \quad I = \int_0^\infty t^{1/4} . e^{-t} . \frac{1}{2\sqrt{t}}\, dt$

$= \frac{1}{2} \int_0^\infty t^{\frac{1}{4}-\frac{1}{2}} . e^{-t}\, dt$

$= \frac{1}{2} \int_0^\infty e^{-t} . t^{-1/2}\, dt$

$= \frac{1}{2} \int_0^\infty e^{-t} . t^{\frac{1}{2}-1}\, dt$

$= \frac{1}{2} \Gamma(1/2)$ Ans

(i) let $I = \int_0^\infty e^{-x^3}\, dx$.

put $x^3 = t \Rightarrow x = t^{1/3}$.

$\Rightarrow dx = \frac{1}{3} \cdot t^{-2/3}\, dt$

As $x \to 0 \Rightarrow t \to 0$ and $x \to \infty \Rightarrow t \to \infty$.

$\therefore \; I = \int_0^\infty e^{-t} \cdot \frac{1}{3} \cdot t^{-2/3}\, dt$.

$= \frac{1}{3} \int_0^\infty e^{-t} \cdot t^{1/3 - 1}\, dt$

$\boxed{I = \frac{1}{3}\,\Gamma(1/3).}$

5 (a) The matrix eqn is given as

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

(i) It will have unique solution if Coefficient matrix is of rank 3. This requires that $\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5-\lambda) \neq 0$

Thus for unique solution $\lambda \neq 5$ and μ may have any value

If $\lambda = 5$, the system of eqn will have no solution for those values of μ for which the matrices

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix} \text{ and augmented matrix } K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$$

are not of the same rank. But A is of rank 2 and K is not of rank 2, unless $\mu = 9$. Thus if $\lambda = 5$ and $\mu \neq 9$ the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have infinite solution

Ans

(b) The characteristic eqn of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley-Hamilton theorem, we have to show that $A^3 - 6A^2 + 9A - 4I = 0$

$$\therefore A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies the theorem.

Now multiplying both sides by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ Ans

6(a) The characteristic eqn of A is given by

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$\therefore \lambda = 1, 2, 3$ are distinct eigenvalues of A.

For $\lambda = 1$, the matrix eqn $[A - I]X = 0$ gives eigenvector

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} -x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ \text{or } x_1 + x_2 = 0 \end{array}$$

The Solution is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda = 2$, eigenvector is given by Solution of $[A - 2I]X = 0$

Thus $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ The Solution is $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda = 3$, eigen vector is obtained from eqn $[A - 3I]X = 0$

Thus $\begin{vmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{vmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The Solution is $X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

Hence the required eigenvectors are $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ Ans

6 (b) The characteristic eqn of A is given by

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \therefore \lambda = 1, 2, 3$$

Since matrix A has three distinct eigenvalues, it has three linearly independent eigenvectors hence it is diagonalizable.

The eigenvector corresponding to $\lambda = 1$, is given by

$$[A - I]X = 0 \Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

The eigenvector corresponding to $\lambda = 2$ is given by

$$[A - 2I]X = 0 \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The eigenvector corresponding to $\lambda = 3$ is given by

$$[A - 3I]X = 0 \Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The Solution is $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Hence the modal matrix $P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

and $P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

$\therefore P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ which is a diagonal matrix

Dey

Or

A square matrix A is said to be orthogonal if

$$A \cdot A^T = A^T A = I$$

If $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

$$\therefore A A^T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence I is orthogonal matrix Ans