MUZAFFARPUR INSTITUTE OF TECHNOLOGY, MUZAFFARPUR, BIHAR – 842003

B. Tech 1th Semester Mid-Term Examination, 2018

Mathematics-I

Time: 2 hours **Branch - ECE+EE** **Full Marks: 20**

Attempt any four questions out of which Question No. 1 is compulsory.

1. Chose the correct option of the following:

(a) If the series $\sum u_n$ is convergent, then $\lim_{n \to \infty} u_n$ is

(i) 0 (ii) ∞ (iii) 1 (iv) -1

(b) Series $\sum \left(\frac{1}{n^{3/2}} \right)$ is

(i) Divergent (ii) Not bounded below (iii) Convergent (iv) None of these.

(c) The improper integral $\int_1^\infty \frac{dx}{x^p}$ converges for

(i) $p < 1$ (ii) $p > 1$ (iii) $p = 1$ (iv) All of the above

(d) $\Gamma(n)\,\Gamma(1-n) =$

(i) π (ii) $\frac{\pi}{\sin n\pi}$ (iii) $\frac{\sin n\pi}{\pi}$ (iv) $\frac{\pi}{\sin n\pi}$

(e) $\Gamma(-3.5) =$

(i) $16\sqrt{\pi}$ (ii) $8\sqrt{\pi}$ (iii) $\frac{16\sqrt{\pi}}{105}$ (iv) $\Gamma(n)$ not defined for $n < 0$

(a)

2. Show that the series $1+r+r^2+r^3+\cdots$ $(r>0)$ converges if $r<1$ and diverges if $r\geq 1$.

Sol^n:— $S_n = 1+r+r^2+\cdots+r^{n-1}$ —— ①

$$S_n = \frac{1-r^n}{1-r} = \frac{1}{1-r} - \frac{r^n}{1-r}$$

Case I:— Suppose $r<1$

$$\lim_{n\to\infty} S_n = \frac{1}{1-r} - \frac{1}{1-r}\lim_{n\to\infty} r^n$$

$$= \frac{1}{1-r} \qquad \left[\because \lim_{n\to\infty} r^n = 0 \text{ if } |r|<1\right]$$

Thus $\langle S_n \rangle$ is a convergent seq^n. so the given series is convergent.

Case II:— Suppose $r=1$

Then $S_n = 1+1+1+\cdots+1 = n$ by ①

clearly, the seq^n $\langle S_n\rangle = \langle n\rangle = \langle 1,2,3,\ldots\rangle$ is divergent.

So, the given series is divergent.

Case III:—

Suppose $r>1$

Then $S_n > n$ by ①

Thus $\lim_{n\to\infty} S_n = +\infty$

So, the series is divergent.

2 (b)

Q:— Test for convergence of the series

$$\sum \left(\sqrt{n^3+1} - \sqrt{n^3}\right).$$

Solⁿ:—

$$u_n = \sqrt{n^3+1} - \sqrt{n^3}$$

$$= \frac{\left(\sqrt{n^3+1} - \sqrt{n^3}\right)\left(\sqrt{n^3+1} + \sqrt{n^3}\right)}{\sqrt{n^3+1} - \sqrt{n^3}}$$

$$= \frac{(n^3+1) - n^3}{\sqrt{n^3+1} - \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1} - \sqrt{n^3}}$$

Let $v_n = \frac{1}{\sqrt{n^3}} = \frac{1}{n^{3/2}}$

Now;

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^{3/2}}{\sqrt{n^3+1} - \sqrt{n^3}}$$

$$= \frac{1}{2} \neq 0 \text{ and finite}$$

So, By limit form test, $\sum u_n$ and $\sum v_n$ converges or diverges together.

Since $\sum v_n = \sum \frac{1}{n^{3/2}}$ is convergent by p-series, where $p > 1$

So the given series $\sum u_n$ is also convergent.

③

Ques :- Evaluate $\int_0^\infty e^{-ax} \cdot x^{m-1} \sin bx \, dx$ in term of Gamma function.

Solution

We have: $\Gamma(m) = \int_0^\infty e^{-x} \cdot x^{m-1} \, dx$.

Put $x = at$

$dx = a\,dt$

As $x \to 0 \Rightarrow t \to 0$

and $x \to \infty \Rightarrow t \to \infty$.

$$\therefore \Gamma(m) = \int_0^\infty e^{-at} \cdot (at)^{m-1} \cdot a\,dt$$

$$= a^m \int_0^\infty e^{-at} \, t^{m-1} \, dt.$$

$$\therefore \int_0^\infty e^{-at} \cdot t^{m-1} \, dt = \frac{\Gamma(m)}{a^m}. \quad \text{———} \quad (1)$$

Now $\because$ $I = \int_0^\infty e^{-ax} x^{m-1} \sin bx \, dx$

$$= \int_0^\infty e^{-ax} \cdot x^{m-1} \cdot (\text{Imaginary Part of } e^{ibx}) \, dx$$

$$= \text{I.P of } \int_0^\infty e^{-(a-ib)x} \cdot x^{m-1} \, dx.$$

$$= \text{I.P of } \frac{\Gamma(m)}{(a-ib)^m}. \quad [\text{using (1)}].$$

$$= \text{I.P of } \frac{\Gamma(m)}{r^m (\cos\theta - i\sin\theta)^m}.$$

$$= \text{I.P of } \frac{\Gamma(m)}{r^m (\cos m\theta - i\sin m\theta)}$$

$$= \text{I.P of } \frac{\Gamma(m)}{r^m (\cos m\theta - i\sin m\theta)} \times \frac{(\cos m\theta + i\sin m\theta)}{(\cos m\theta + i\sin m\theta)}.$$

$$= \text{I.P of } \frac{\Gamma(m) \cdot (\cos m\theta + i\sin m\theta)}{r^m (\cos^2 m\theta + \sin^2 m\theta)}.$$

$$= \text{I.P of } \frac{\Gamma(m) \cdot (\cos m\theta + i\sin m\theta)}{r^m}.$$

$$= \frac{\Gamma(m) \cdot \sin m\theta}{r^m}$$

(4) **Ques:** Evaluate the following improper integrals

(ii) $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$ (i) $\int_0^\infty e^{-x^3} dx$.

in terms of Gamma functions.

Soln. (ii) $\int_0^\infty \sqrt{x} \cdot e^{-x^2} dx$.

put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{1}{2\sqrt{t}} dt$.

As $x \to 0 \Rightarrow t \to 0$ and $x \to \infty \Rightarrow t \to \infty$

$\therefore$ $I = \int_0^\infty t^{1/4} \cdot e^{-t} \cdot \frac{1}{2\sqrt{t}} dt$

$= \frac{1}{2}\int_0^\infty t^{\frac{1}{4}-\frac{1}{2}} \cdot e^{-t} dt$

$= \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{-1/2} dt$

$= \frac{1}{2}\int_0^\infty e^{-t} \cdot t^{\frac{1}{2}-1} dt$

$= \frac{1}{2}\Gamma(1/2)$ Ans

(i) let $I = \int_0^\infty e^{-x^3}\, dx$.

put $x^3 = t \Rightarrow x = t^{1/3}$.

$\Rightarrow dx = \frac{1}{3} \cdot t^{-2/3}\, dt$

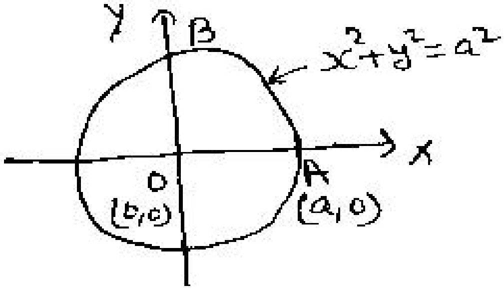
As $x \to 0 \Rightarrow t \to 0$ and $x \to \infty \Rightarrow t \to \infty$.

$\therefore I = \int_0^\infty e^{-t} \cdot \frac{1}{3} \cdot t^{-2/3}\, dt$.

$= \frac{1}{3} \int_0^\infty e^{-t} \cdot t^{1/3 - 1}\, dt$

$$I = \frac{1}{3}\Gamma\left(\frac{1}{3}\right).$$

Q. 5 (a) The region of integration R is bounded by $x^2+y^2=a^2$.



Let us integrate first w.r.t y and then w.r.t x.
The limits of OAB are x varies from $x=0$ to $x=a$ and y varies from $y=0$ to $y=\sqrt{a^2-x^2}$

$$\therefore \iint_R xy\,dx\,dy = \int_{x=0}^{a} dx \int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy = \int_0^a x\left|\frac{y^2}{2}\right|_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x(a^2-x^2)}{2}\,dx = \frac{1}{2}\int_0^a (a^2x - x^3)\,dx$$

$$= \frac{1}{2}\left|\frac{a^2x^2}{2} - \frac{x^4}{4}\right|_0^a = \frac{1}{2}\left|\frac{a^4}{2} - \frac{a^4}{4}\right| = \frac{a^4}{8}$$ Ans

(b) $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz$

Since the limits of inner integral are function of x and y so, they are limits for z. we first integrate w.r.t z.

$$I = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} dz\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1-x} |z|_0^{1-x-y}\,dy\,dx$$

$$= \int_{x=0}^{1} dx \int_{y=0}^{1-x} (1-x-y)\,dy = \int_0^1 \left|y - xy - \frac{y^2}{2}\right|_0^{1-x} dx$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{1}{2}(1-x)^2\right] dx$$

$$= \frac{1}{2}\int_0^1 (1-2x+x^2)\,dx = \frac{1}{2}\left|x - \frac{2x^2}{2} + \frac{x^3}{3}\right|_0^1 = \frac{1}{2}\left|1-1+\frac{1}{3}\right|$$

$$= \frac{1}{6}$$ Ans