MUZAFFARPUR INSTITUTE OF TECHNOLOGY, MUZAFFARPUR, BIHAR – 842003

(Under the department of Science & Technology, Bihar, Patna)

B.Tech 3rd Semester Mid-Term Examination, 2018

Mathematics-III

Time: 2 hours **Full Marks: 20**

Attempt any four questions out of which Question No. 1 is compulsory.

1. Chose the correct option of the following:

   (a) The series solution of Legendre differential equation about the origin will converge only if

   (i) $|x| = 1$ (ii) $|x| < 1$ (iii) $|x| \geq 1$ (iv) None of these

   (b) If A function $f(z)$ of the complex variable is given as $f(x,y) = u(x,y) + iv(x,y)$, where $u(x,y) = 2kxy$ and $v(x,y) = x^2 - y^2$. The value of k, for which the function is analytic, is

   (i) 0 (ii) ∞ (iii) 1 (iv) -1

   (c) An analytic function of a complex variable is given as $f(x,y) = u(x,y) + iv(x,y)$. If $u(x,y) = x^2 - y^2$, then expression for $v(x,y)$ in terms of x, y and a general constant C would be

   (i) $xy + C$ (ii) $\frac{x^2 + y^2}{2} + C$ (iii) $2xy + C$ (iv) $\frac{(x-y)^2}{2} + C$

   (d) Solution of the PDE $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ is

   (i) $z = \phi_1(y + 2x) + \phi_2(y - x)$ (ii) $z = \phi_1(y) + \phi_2(x)$

   (iii) $z = \phi_1(y + x) + \phi_2(y - x)$ (iv) $z = x\phi_1(y)$

   (e) If a fair coin is tossed four times. What is the probability that two heads and two tails will results?

   (i) 3/8 (ii) 1/2 (iii) 5/8 (iv) 3/4

P.T.O

②

Q:- A tightly stretched string .... the string at any time $t>0$.

Sol^n:- The wave eq^n is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— ①

The sol^n of ① is

$$y(x,t) = (c_1 \cos cpt + c_2 \sin cpt)(c_3 \cos px + c_4 \sin px) \quad \text{—— ②}$$

Boundary conditions are

$$y(0,t)=0, \quad y(l,t)=0$$

Initial conditions are

$$\frac{\partial y}{\partial t} = 0 \text{ at } t=0 \text{ and } y(x,0) = \mu x(l-x)$$

From ②, $y(0,t)=0 = (c_1 \cos cpt + c_2 \sin cpt) c_3$

$\Rightarrow c_3 = 0$

$\therefore y(x,t) = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin px$ —— ③

$y(l,t)=0 = (c_1 \cos cpt + c_2 \sin cpt) c_4 \sin pl$

$\Rightarrow \sin pl = 0 = \sin n\pi \quad (n \in I)$

$\Rightarrow p = \frac{n\pi}{l}$

From ③, $y(x,t) = \left(c_1 \cos \frac{n\pi ct}{l} + c_2 \sin \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$ —— ④

Now, $\frac{\partial y}{\partial t} = \frac{n\pi c}{l}\left(-c_1 \sin \frac{n\pi ct}{l} + c_2 \cos \frac{n\pi ct}{l}\right) c_4 \sin \frac{n\pi x}{l}$

At $t=0$, $\frac{\partial y}{\partial t}=0=\frac{n\pi c}{l}c_2 c_4 \sin\frac{n\pi x}{l}$

$\Rightarrow c_2=0$

From (4), $y(x,t)=c_1 c_4 \cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$

$\Rightarrow y(x,t)=b_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$, where $c_1 c_4=b_n$

The most general solⁿ is

$$y(x,t)=\sum_{n=1}^{\infty} b_n\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l} \quad \text{——— (5)}$$

$$y(x,0)=\mu(lx-x^2)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

where $b_n=\frac{2}{l}\int_0^l \mu(lx-x^2)\sin\frac{n\pi x}{l}\,dx$

$$=\frac{2\mu}{l}\left[-(lx-x^2)\frac{l}{n\pi}\cos\frac{n\pi x}{l}+(l-2x)\left(\frac{l}{n\pi}\right)^2\sin\frac{n\pi x}{l}+(-2)\left(\frac{l}{n\pi}\right)^3\cos\frac{n\pi x}{l}\right]_0^l$$

$$=\frac{2\mu}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n+\frac{2l^3}{n^3\pi^3}\right]$$

$$=\frac{2\mu}{l}\frac{2l^3}{n^3\pi^3}\left[-(-1)^n+1\right]=\frac{4\mu l^2}{n^3\pi^3}\left[-(-1)^n+1\right]$$

So, From (5), we have

$$y(x,t)=\frac{4\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[-(-1)^n+1]}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

$$=\frac{4\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^3}\cos\frac{n\pi ct}{l}\sin\frac{n\pi x}{l}$$

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3 (a)

Q:- Solve $x^2p+y^2q=(x+y)z$

Solⁿ:- Lagrange's auxiliary eqⁿ is

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z}$$

Taking I^{st} and II^{nd} terms

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Now, integrating, we get

$$-\frac{1}{x} = -\frac{1}{y} + c_1 \quad \text{--- (1')}$$

$$\frac{1}{y} - \frac{1}{x} = c_1 \quad \text{--- (2)}$$

Now, let

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx - dy}{x - y} = \frac{dz}{z}$$

$$\frac{d(x-y)}{x-y} = \frac{dz}{z}$$

~~Integrating~~ Integrating on both sides, we have

$$\log(x-y) = \log z + \log c_2$$

or $$\frac{x-y}{z} = c_2 \quad \text{--- (3)}$$

Hence the complete solⁿ is

$$f\left(\frac{1}{y} - \frac{1}{x}\right) = \frac{x-y}{z}.$$

3 (b)

Q:— Form the p.d. eqⁿ from the relation

$$z = f(x^2 - y^2) \quad \text{--- (1)}$$

Solⁿ:— Partially diff. eqⁿ (1) w.r.t. x, we get

$$p = \frac{\partial z}{\partial x} = 2x f'(x^2 - y^2) \quad \text{--- (2)}$$

Now, partially diff. eqⁿ (1) w.r.t. y, we get

$$q = \frac{\partial z}{\partial y} = -2y f'(x^2 - y^2) \quad \text{--- (3)}$$

Dividing ② by ③, we have

$$\frac{p}{q} = -\frac{x}{y}$$

$$\boxed{yp + xq = 0}$$

4 ⓐ

Q:- Show that the funⁿ $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although the C-R eqⁿˢ. are satisfied at that point.

Solⁿ:- Let $f(z) = u(x,y) + iv(x,y) = \sqrt{|xy|}$

So that $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$

we have, at the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0.$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{0-0}{x} = 0, \quad \frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

Hence the Cauchy-Riemann equations are satisfied at the origin.

Now $$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|} - 0}{x+iy}$$

Letting $z \to 0$ along $y = mx$, we get

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x(1+im)} = \frac{\sqrt{|m|}}{(1+im)}$$

This limit is not unique since it depends on m.

Hence $f'(0)$ does not exist and so $f(z)$ is not analytic at $z = 0$.

(4) (b) Given $u = e^{-x}(y\cos y - x\sin y)$

$$\frac{\partial u}{\partial x} = e^{-x}(-\sin y) - e^{-x}(y\cos y - x\sin y)$$

$$\frac{\partial u}{\partial x} = -e^{-x}(\sin y + y\cos y - x\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(\sin y + y\cos y - x\sin y) - e^{-x}(-\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(2\sin y + y\cos y - x\sin y)$$

Lly: $\frac{\partial u}{\partial y} = e^{-x}(\cos y - y\sin y - x\cos y)$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-\sin y - y\cos y - \sin y + x\sin y)$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

and hence $u(x,y) = e^{-x}(y\cos y - x\sin y)$ is harmonic function.

Now for conjugate harmonic $v(x,y)$.

By differential eqn method, we have;

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy \quad \text{--- (1)}$$

Since $f(z) = u + iv$ is analytic
$\Rightarrow f(z)$ satisfies C-R eqn
$\Rightarrow u_x = v_y$ and $v_x = -u_y$

Since R.H.S of (1) is exact differential eqn.

Therefore the solution of (1) is given by

$$\int dv = \int \left(-\frac{\partial u}{\partial y}\right) dx + \int \left(\frac{\partial u}{\partial x}\right) dx + c$$

treating y as constant

only those terms independent of x

$$\Rightarrow \int dv = \int -e^{-x} (\cos y - y\sin y - x\cos y)\, dx$$

$$+ \int 0\, dx + c$$

$$= -\int e^{-x} \cos y\, dx + y\sin y \int e^{-x} dx + \cos y \int x e^{-x} dx + c$$

$$= +\cos y\, e^{-x} - y\sin y\, e^{-x} + \cos y\left[-x e^{-x} - e^{-x}\right] + c$$

$$= e^{-x}(\cos y - y\sin y - x\cos y - \cos y) + c$$

$$\boxed{v = -e^{-x}(y\sin y + x\cos y) + c}$$

and hence

$$f(z) = u(x,y) + i v(x,y)$$

$$= e^{-x}(y\cos y - x\sin y) + i\left[-e^{-x}(y\sin y + x\cos y) + c\right]$$

$$= e^{-x}(y\cos y - x\sin y) - i e^{-x}(y\sin y + x\cos y) + K$$

where $K = ic$

(5) (a) Find the value of $J_4(x)$ ~~and~~ in terms of $J_0(x)$.

Soln. From Recurrence Relation; we have

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad \text{———} (1)$$

put $n = 1, 2, 3$ in equation (1); we get

for $n=1$; $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

for $n=2$; $J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$

for $n=3$; $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

$$= \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \frac{24}{x^2}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - \frac{6}{x} J_1(x) - \frac{2}{x} J_1(x) + J_0$$

$$= \frac{48}{x^3} J_1(x) - \frac{24}{x^2} J_0(x) - \frac{6}{x} J_1(x) - \frac{2}{x} J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

(5)(b) Prove that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$

__Proof__ As we have the solution of Bessel's equation is as

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \lfloor m \, \Gamma(m+n+1)}$$

So;

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{d}{dx}\left[\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2n}}{2^{2m+n} \lfloor m \, \Gamma(m+n+1)}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2n) x^{2m+2n-1}}{2^{2m+n} \lfloor m \, (m+n)\Gamma(m+n)}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m 2\cancel{(m+n)} x^{2m+(n-1)}}{2^{2m+n} \lfloor m \, \cancel{(m+n)}\Gamma(m+n)}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+(n-1)}}{2^{2m+(n-1)} \lfloor m \, \Gamma[m+(n-1)+1]}$$

$$\boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}$$

Proved

6. (a) There are two cases of transferring a ball from A to B.

<u>Case I</u> :- When a white ball is transferred from A to B then $P(\text{transfer of a white ball}) = \frac{2}{6} = \frac{1}{3}$

After transfer of a white ball, Bag B contains 6 white & 7 black balls. Hence P(Drawing a white ball from B)

$= P(\text{transfer of a white ball}) \times P(\text{Drawing a white ball})$

$= \frac{1}{3} \times \frac{6}{13} = \frac{2}{13}$

<u>Case II</u> :- When a black ball is transferred from A to B

$P(\text{transfer of a black ball}) = \frac{4}{6} = \frac{2}{3}$

After transfer of a black ball, Bag B contains 5 white, 8 black balls.

$\therefore$ P(Drawing a white ball from bag B after transfer)

$= \frac{2}{3} \times \frac{5}{13} = \frac{10}{39}$

Therefore, the req. probability = either Case I or Case II

$= \frac{2}{13} + \frac{10}{39} = \frac{16}{39}$ Ans

(b) If $f(x)$ is a density function, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} e^{-x}\,dx = 1$$

Hence the function $f(x)$ satisfies the requirement for density function.

(i) Required probability $= P(1 \leq x \leq 2)$

$$= \int_{1}^{2} e^{-x}\,dx = e^{-1} - e^{-2} = 0.233 \text{ Ans}$$

(ii) The Cumulative prob. function $F(x) = \int_{-\infty}^{x} f(x)dx$

$$\therefore F(2) = \int_{-\infty}^{2} f(x)dx = \int_{-\infty}^{0} 0.dx + \int_{0}^{2} e^{-x}dx$$

$$= 1 - e^{-2} = 1 - 0.135 = 0.865 \text{ Ans}$$

7(a) m.g.f of X is $M_0(t) = E(e^{tx}) = \sum_{x=1}^{\infty} e^{tx} p(x)$ By defn

$$= \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right] = \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1}$$

$$\therefore M_0(t) = \frac{e^t}{2}\left[\frac{1}{1 - \frac{e^t}{2}}\right] = \frac{e^t}{2 - e^t} \text{ Ans}$$

Mean $\mu_1' = \left|\frac{d}{dt} M_0(t)\right|_{t=0} = \left|\frac{d}{dt} \frac{e^t}{2-e^t}\right|_{t=0} = \left|\frac{2e^t}{(2-e^t)^2}\right|_{t=0}$

$$= \frac{2 \cdot 1}{1} = 2 \text{ Ans}$$

variance $= \mu_2' - (\mu_1')^2$, where $\mu_2' = \left|\frac{d^2}{dt^2} M_0(t)\right|_{t=0}$

$$= \left|\frac{8e^t - 2e^{2t}}{(2-e^t)^3}\right|_{t=0} = \frac{8-2}{1} = 6$$

variance $= 6 - (2)^2 = 2$

$\therefore$ S.D. $= \sqrt{2}$ Ans

7(b) Let p be the prob. of hitting the target.

Given that $p = \frac{1}{3}$ $\therefore q$ = prob. of no hit $= 1 - p = \frac{2}{3}$

(i) Let X be number of hits and $n = 5$

$$\therefore P(X \geq 2) = 1 - [P(X=0) + P(X=1)] = 1 - \left[{}^5C_0\left(\tfrac{1}{3}\right)^0\left(\tfrac{2}{3}\right)^5 + {}^5C_1\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^4\right]$$

$$= 1 - \left(\tfrac{2}{3}\right)^5 - 5\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)^4 = \frac{131}{243} = 0.539 \text{ Ans}$$

(ii) Prob. of at least one hit = 1 − prob. of no hit out of n

$$= 1 - q^n = 1 - \left(\tfrac{2}{3}\right)^n$$

Given that $1-\left(\frac{2}{3}\right)^n > 0.9$

$\Rightarrow 0.1 > \left(\frac{2}{3}\right)^n \Rightarrow 0.1 \times 3^n > 2^n$

$\Rightarrow n = 6$

Thus the smallest value of $n = 6$ for which the prob. of hitting the target atleast once is more than 90%

Ans