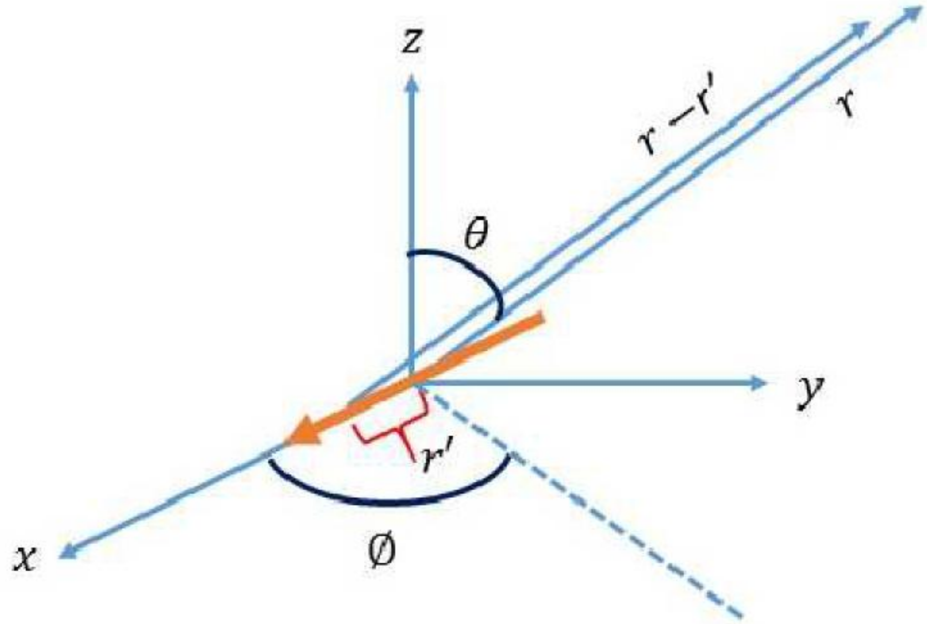
MID SEMESTER EXAM-2018 SOLUTIONS
7th SEMESTER
Advance Electromagnetic Field Theory (041709)

1. A horizontal infinitesimal electric dipole of constant current I_0 is placed symmetrically about the origin and directed along x-axis. The components of magnetic field in far field will be?

Solution: The arrangement of the current element is shown in figure given below.



As it is mentioned that the current is x directed, therefore, the vector magnetic potential will also be x-directed and it will be symmetric about θ and φ. Hence the vector magnetic potential in the far field will be

$$A_x(\mathrm{r}) = (I_0 \mathrm{dl}/4\pi \mathrm{r})\, e^{-jk_0 r}$$

It is worthy to note that, we have assumed that the current element radiates in free space, therefore, the vector magnetic potential, ˜ A and magnetic field intensity, H are related to each other by, H = ∇ × A. As the vector magnetic potential is in the Cartesian co-ordinate system, we need to convert it into the Spherical co-ordinate system. This can be achieved by using the following matrix formula.

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\Phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\Phi & \sin\theta\sin\Phi & \cos\theta \\ \cos\theta\cos\Phi & \cos\theta\sin\Phi & -\sin\theta \\ -\sin\Phi & \cos\Phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Now, the components of the vector magnetic potential in spherical coordinate system will be,

$$A_r = A_x \sin\theta\cos\Phi$$
$$= (I_0 \mathrm{dl}/4\pi \mathrm{r})\, e^{-jk_0 r} \sin\theta\cos\Phi$$

$$A_\theta = A_x \cos\theta\cos\Phi$$
$$= (I_0 \mathrm{dl}/4\pi \mathrm{r})\, e^{-jk_0 r} \cos\theta\cos\Phi$$
$$A_\Phi = A_x(-\sin\Phi)$$
$$= (I_0 \mathrm{dl}/4\pi \mathrm{r})\, e^{-jk_0 r} (-\sin\Phi)$$

Now, from the curl equation, we can easily determine the components of the magnetic fields in far field. Writing the curl equation in spherical coordinate system

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta a_\varphi \\ \partial / \partial r & \partial / \partial \theta & \partial / \partial \phi \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}$$

Solving the curl equation given above we get,

$$\begin{aligned} \tilde{H}_r &= \frac{1}{r^2 \sin\theta}[\frac{\partial}{\partial \theta}\{r \sin\theta \tilde{A}_\phi\} - \frac{\partial}{\partial \phi}\{r \tilde{A}_\theta\}] \\ &= \frac{1}{r^2 \sin\theta}[\frac{\partial}{\partial \theta}\{r \sin\theta \frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r}(-\sin\phi)\} - \frac{\partial}{\partial \phi}\{r \frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r}\}] \\ &= \frac{1}{r^2 \sin\theta}[-\sin\phi \cos\theta \frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} - (-\sin\phi)\frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} \cos\theta] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \tilde{H}_\theta &= -\frac{1}{r^2 \sin\theta} \cdot r[\frac{\partial}{\partial r}\{r \sin\theta \tilde{A}_\phi - \frac{\partial}{\partial \phi}\tilde{A}_r\}] \\ &= -\frac{1}{r \sin\theta}[\frac{\partial}{\partial r}\{r \sin\theta \frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r}(-\sin\phi)\} - \frac{\partial}{\partial \phi}\{\frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} \sin\theta \cos\phi\}] \\ &= -\frac{1}{r \sin\theta}[(-jk_0)\sin\theta(-\sin\phi)\frac{\tilde{I}_0 dl}{4\pi} e^{-jk_0 r} - \cancelto{0}{\frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} \sin\theta(-\sin\phi)}\quad] \\ &= -\frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} jk_0 \sin\phi \end{aligned}$$

and

$$\begin{aligned} \tilde{H}_\theta &= \frac{1}{r^2 \sin\theta} \cdot r\sin\theta[\frac{\partial}{\partial}\{r \frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} \cos\theta \cos\phi\} - \frac{\partial}{\partial}\{\frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} \sin\theta \cos\phi\}] \\ &= \frac{1}{r}[\frac{\tilde{I}_0 dl}{4\pi} e^{-jk_0 r}(-jk_0)\cos\theta \cos\phi - \cancelto{0}{\frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r} \cos\theta \cos\phi}\quad] \\ &= -\frac{\tilde{I}_0 dl}{4\pi r} e^{-jk_0 r}(-jk_0 \cos\theta \cos\phi) \end{aligned}$$

2. a) A z-directed infinitesimally small dipole of length λ/50 is operating at 30 MHz. Determine the minimum distance from the antenna where the angular distribution of the fields are independent of the distance from the antenna.

Solution: The minimum distance from the antenna where the angular distribution of the fields are independent of the distance from the antenna indicates the beginning of the far-filed regions. The lower boundary of the far field region can be determined by using R = $2D^2/\lambda$. Where, D indicates the maximum dimension of the antenna.

$$\begin{aligned} \text{So, } R &= 2D^2/\lambda \\ &= 2\lambda^2/(50^2 \times \lambda) \\ &= 2c/(f \cdot 50^2) \\ &= 2 \times 3 \times 10^8/(50^2 \times 30 \times 10^6) \\ &= 8 \times 10^{-3} \text{ m} \\ &= 8 \text{ mm} \end{aligned}$$

b) An infinitesimally small dipole is z−directed and fed by an uniform current of I = 0.5sin(ω_0t).The radiated power density in the far field is given by $W_{rad} = B_0\ sin^2\theta/r^2\ \cdot a_r$ Watts/$meter^2$.Compute the radiation resistance of the antenna in the far field.

Solution: The radiated power density of the antenna is given as $W_{rad} = B_0\ sin^2\theta/r^2\ \cdot a_r$ Watts/$meter^2$. So, the total radiated power over a spherical surface or radius 'r' can be determined by performing an integration over the surface. Thus,

$$P_{rad} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{B_0\sin^2\theta}{r^2}r^2\sin\theta d\theta d\phi$$

$$= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} B_0\sin^3\theta d\theta d\phi$$

$$= \frac{2\pi B_0}{4}[\int_{\theta=0}^{\pi} 3\sin\theta d\theta - \int_{\theta=0}^{\pi}\sin 3\theta d\theta]$$

$$= \frac{2\pi B_0}{4}[6-\frac{2}{3}]$$

$$= \frac{8\pi B_0}{3}$$

As. the current in the dipole is sinusoidal, hence, its r.m.s value can be easily calculated as $\frac{I_0}{\sqrt{2}} = \frac{0.5}{\sqrt{2}}$ = 0.36A. So, the radiation resistance, radiated power and the r.m.s value of the current are related to each other by $P_{rad} = |Irms|^2\ \cdot R_{rad}$. Thus,

$$R_{rad} = \frac{P_{rad}}{|I_{rms}|^2}$$

$$= \frac{8\pi B_0}{0.36^2\cdot 3}$$

$$= 20.57\pi B_0 \quad \Omega$$

3. a) An antenna is radiating in an lossless, non-magnetic medium of unknown dielectric constant. It has a far-field electric field given by$E_\theta = (\frac{2.12}{r})e^{-jkr}\sin\theta$. If it radiates a total power of 200 mW, determine the dielectric constant of the medium.

Solution: Let us assume the wave impedance of the unknown, lossless and non-magnetic medium is η. In the far-filed, the wave are of TEM^r types. Hence, the magnetic field at the far-field will be $H_\Phi = \frac{E_\theta}{\eta} = \frac{2.12}{\eta}$r e^{-jkr} sin θ. Thus, total radiated power by the antenna can be calculated by computing the Poynting vector. So,

$$P_{rad} = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{2\cdot 12^2}{2\eta r^2} r^2 \sin\theta \cdot \sin^2\theta \, d\theta \, d\phi$$

$$= \frac{(2\cdot 12)^2}{2\eta}\int_0^{\pi}\int_0^{2\pi} \sin^3\theta \, d\theta \, d\phi$$

$$= \frac{4.5\times\pi}{\eta}\int_0^{\pi} \sin^3\theta \, d\theta = \frac{4.5\pi}{8\eta}\times\frac{1}{\cancel{4}}\times\cancel{16}^{\,4}$$

Given, $P_{rad} = 200\ \text{mW} = \frac{6\pi}{\eta}$

$$\eta = \frac{6\pi\times10^3}{200} = 30\pi$$

$$\frac{120\pi}{\sqrt{\epsilon_r}} = 30\pi$$

$$\therefore \epsilon_r = 16$$

b) A certain antenna with an efficiency of 95% has maximum radiation intensity of 0.5 W/Steradian. Determine its directivity when the input power is 0.4 W.

Solution:

The directivity can be mathematically calculated by

$$D = 4\pi\frac{\text{Radiation Intensity}}{\text{Total Radiated Power}}$$

Now, it has been mentioned that the antenna has an efficiency of 95%. Hence, the total power radiated by the specified antenna is $P_{rad} = 0.95\times P_{in} = 0.95\times 0.4 = 0.38$ Watts.

SO, the calculated directivity from the above formula $D = 4\pi\frac{0.5}{0.38} = 16.53$.

4. An air-filled rectangular waveguide has cross-sectional dimensions *a* = 6 cm and *b* = 3 cm. Given that, $Ez = 5\sin(2\pi x/a)\sin(3\pi y/b)\cos(10^{12}t-\beta z)$ A/m the intrinsic impedance of this mode and the average power flow in the guide are

Solution:

$E_z \neq 0$, This must be TM_{23} mode ($m=2$, $n=3$)

$$f_c = \frac{1}{2\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2} = \frac{3\times10^8}{2\times10^{-2}}\sqrt{\frac{4}{36}+\frac{9}{9}} = 15.81\,GHz$$

$$f = \frac{\omega}{2\pi} = \frac{10^{12}}{2\pi} = 159.15\,GHz$$

$$\eta_{TM} = \eta_0\sqrt{1-(f_c/f)^2} = 377\times\sqrt{1-\left(\frac{15.81}{159.15}\right)^2}$$

$$\boxed{\eta_{TM} = 375.1\,\Omega}$$

$$P_{avg} = \frac{|E_{xs}|^2+|E_{ys}|^2}{2\eta_{TM}}\,a_z \quad W/m^2$$

$$= \frac{\beta^2 E_0^2}{2h^4\eta_{TM}}\left[\left(\frac{2\pi}{a}\right)^2\cos^2\left(\frac{2\pi}{a}x\right)\sin^2\left(\frac{3\pi}{b}y\right)+\left(\frac{3\pi}{b}\right)^2\sin^2\left(\frac{2\pi}{a}x\right)\cos^2\left(\frac{3\pi}{b}y\right)\right]a_z$$

$$P_{ave} = \int P_{avg}\cdot ds = \int_{x=0}^{a}\int_{y=0}^{b} P_{avg}\cdot dx\,dy\,a_z$$

$$= \frac{\beta^2 E_0^2}{2h^4\eta_{TM}}\cdot\frac{1}{4}\left[\frac{4\pi^2}{a^2}+\frac{9\pi^2}{b^2}\right]ab = \frac{\beta^2 E_0^2}{8h^2\eta_{TM}}\,ab$$

$$\beta = \frac{\omega}{c}\sqrt{1-(f_c/f)^2} = 3.317\times10^3$$

$$h^2 = \frac{4\pi^2}{a^2}+\frac{9\pi^2}{b^2} = 1.098\times10^5$$

$$P_{avg} = \frac{(3.317)^2\times10^6\times25\times6\times3\times10^{-4}}{8\times1.098\times10^5\times375.1} = 1.5\,mW$$

5. The longitudinal electric field for TM_{11} mode is given by $E_z = \sin(5x)\ \sin(8y)e^{-j\beta z}$ V/m. Derive remaining field components and dispersion relation of the mode. Find the cut-off frequency and phase velocity of the mode at a frequency twice the cut-off frequency of the mode.

Solution:

The dispersion relation can be obtained by substituting in the wave equation,

$$\nabla^2 E_z + \omega^2 \mu\epsilon E_z = 0$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} + \omega^2\mu\epsilon E_z = 0$$

$$-25 - 64 - \beta^2 + \omega^2\mu\epsilon = 0$$

$$\therefore \beta = \sqrt{\omega^2\mu\epsilon - 89}$$

So, we get $h = \sqrt{89}$

Since, the mode is TM, $H_z = 0$, the other field components can be written as,

$$E_x = \frac{-j\beta}{h^2}\frac{\partial E_z}{\partial x} = \frac{-j\beta}{89} 5\cos(5x)\sin(8y)e^{-j\beta z}$$

$$E_y = \frac{-j\beta}{h^2}\frac{\partial E_z}{\partial y} = \frac{-j\beta}{89} 8\sin(5x)\cos(8y)e^{-j\beta z}$$

$$H_x = \frac{j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial y} = \frac{j\omega\epsilon}{89} 8\sin(5x)\cos(8y)e^{-j\beta z}$$

$$H_y = \frac{-j\omega\epsilon}{h^2}\frac{\partial E_z}{\partial x} = \frac{-j\omega\epsilon}{89} 5\cos(5x)\sin(8y)e^{-j\beta z}$$

The cut-off frequency of the mode is

$$\omega_c = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{89} = 3\times10^8 \times \sqrt{89} = 2.83\times10^9 \text{ rad/s}$$

$$f_c = 4.50\times10^8 \text{ Hz}$$

Phase velocity, $v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu\epsilon - 89}} = 3.46\times10^8$ m/s

Group velocity, $v_g = \frac{\partial\omega}{\partial\beta} = \frac{1}{\mu\epsilon}\frac{\beta}{\omega} = 2.598\times10^8$ m/s