Department of Electrical Engineering

7^{th} semester (E)

Subject - Linear Control Theory.

Q.1 (a) Rise time — It is the time taken by the step response to reach 100 % of the final response in first attempt. (denoted by t_r)

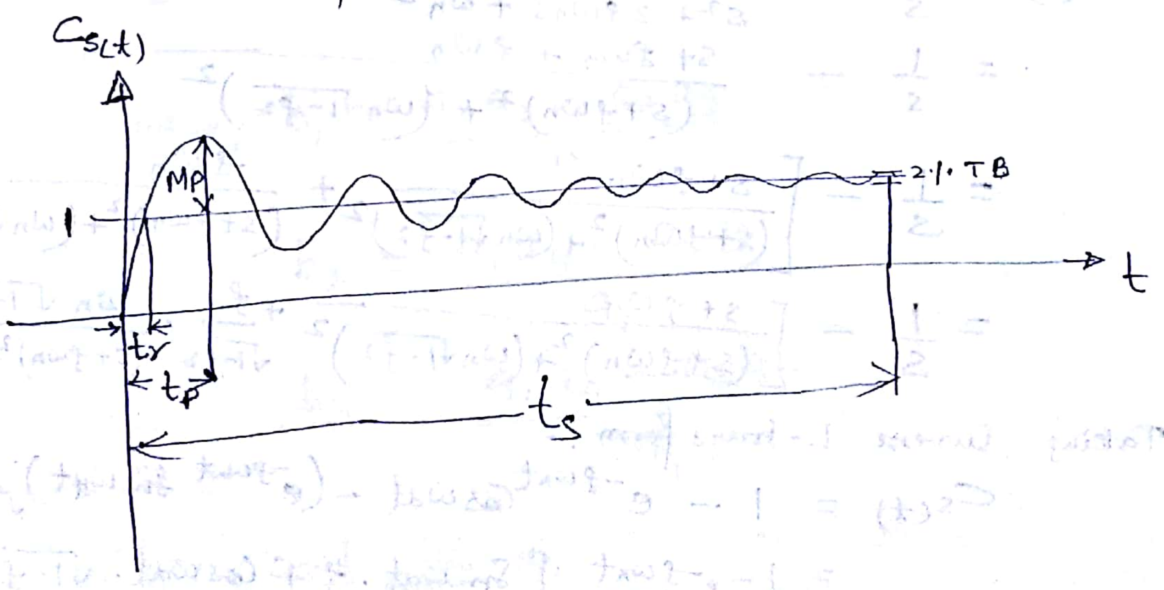
Peak time — It is the time taken by the step response to reach the peak of the step response. (denoted by t_p)

Peak overshoot — It is the normalised difference between the peak response and the final response (denoted by M_p)

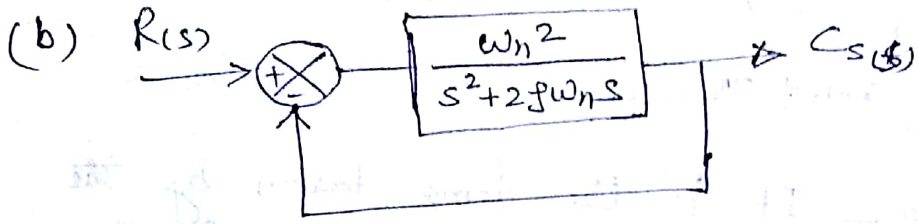
$$M_p = \frac{C(t_p) - C(\infty)}{C(\infty)}$$

Settling time — It is the time taken by the step response to reach and stay within a specified tolerance band of its final response (denoted by t_s)

2 % tolerance band for accurate analysis.
5 % " " " rough analysis.



(b)



The above figure represents UFCS with second-order system.

$$TF = \frac{\frac{\omega_n^2}{s^2+2\zeta\omega_n s}}{1+\frac{\omega_n^2}{s^2+2\zeta\omega_n s}} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Here, $R(s) = 1/s$

$$C_s(s) = (TF)R(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} =$$

$$= \frac{A}{s} + \frac{B.s+C}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{As^2+2\zeta\omega_n A s+A\omega_n^2+Bs^2+Cs}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

Equating the coefficients of different s^n:-

$$A.\omega_n^2 = \omega_n^2 \Rightarrow A = 1$$

$$A.2\zeta\omega_n + C = 0 \Rightarrow C = -2\zeta\omega_n$$

$$A+B = 0 \Rightarrow B = -A = -1$$

$$C_s(s) = \frac{1}{s} - \frac{(s+2\zeta\omega_n)}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+\zeta\omega_n+\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2}$$

$$= \frac{1}{s} - \left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2} + \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2}\right]$$

$$= \frac{1}{s} - \left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2} + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\frac{\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2+(\omega_n\sqrt{1-\zeta^2})^2}\right]$$

Taking inverse L-transform :-

$$C_s(t) = 1 - e^{-\zeta\omega_n t}\cos\omega_d t - (e^{-\zeta\omega_n t}\sin\omega_d t)\frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\omega_d t.\zeta + \cos\omega_d t.\sqrt{1-\zeta^2}\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\omega_d t.\cos\phi + \cos\omega_d t.\sin\phi\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\phi)\ ;\ \phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

(2) (a) Routh stability criterion – It states that for a system to be stable it is necessary and sufficient that each term of the first column of Routh array of its characteristic equation be positive if $a_0 > 0$ [C.E. $a_0 s^n + \cdots + a_n = 0$] If the condition is not met then system is unstable and the no. of sign changes in the first column of the Routh array corresponds to the no. of roots of C.E. lying in RHP.

Let C.E. be $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n = 0$

The corresponding Routh array is :-

$$\begin{array}{c|cccc} s^n & a_0 & a_2 & a_4 & a_6 \\ s^{n-1} & a_1 & a_3 & a_5 & a_7 \\ s^{n-2} & b_1 & b_2 & b_3 & \\ \vdots & & & & \\ s^0 & & & & \end{array}$$

where, $b_1 = \dfrac{a_1 a_2 - a_0 a_3}{a_1}$

$b_2 = \dfrac{a_1 a_4 - a_0 a_5}{a_1}$

$b_3 = \dfrac{a_1 a_6 - a_0 a_7}{a_1}$

and so on.

(b) The characteristic equation is:

$$q(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

Corresponding Routh array is

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0
s^2				
s^1				
s^0				

It can be observed that s^3-row is all-zero row

So auxiliary polynomial $A(s) = 2s^4 + 12s^2 + 16$

Differentiating $A(s)$ once w.r.to s we get:-

$$A'(s) = 8s^3 + 24s$$
$$= 8(s^3 + 3s)$$

So, the new coefficients of s^3-row are $(1, 3)$

and new Routh array is:

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	1	3	0	0
s^2	6	16	0	0
s^1	1/3	0	0	0
s^0	16			

As there is no sign change in the first column and also this problem belongs to all-zero row case, so the system is marginally stable.

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(3) Poles at $s = 0$

$= -2$

$= \frac{-6 \pm \sqrt{36-100}}{2} = \frac{-6 \pm j8}{2} = -3 \pm j4$

Centroid: $\sigma_A = \frac{[(-2)+(-3)+(-3)] - [0]}{4-0} = -2$

Angle of asymptotes: $\phi_A = \frac{(2q+1)180^\circ}{P-Z}$; $q = 0, 1, 2 \ldots$

$= \frac{(2q+1)180^\circ}{4-0} = (2q+1)\,45^\circ = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

Breakaway points :- C.E. is :- $1 + G(s) = 0$

$s(s+2)(s^2+6s+25) + K = 0$

$-K = (s^2+2s)(s^2+6s+25)$

$= s^4 + 8s^3 + 37s^2 + 50s$

$-\frac{dK}{ds} = 4s^3 + 24s^2 + 74s + 50$

Setting $\frac{dK}{ds} = 0 \Rightarrow 4s^3 + 24s^2 + 74s + 50 = 0$

$\Rightarrow s = -0.9 , \; -2.55 \pm j2.7$

$s = -0.9$ only feasible breakaway point

Immediate Breakaway direction :

$\theta = \pm \frac{180^\circ}{r} = \pm 90^\circ$

Intersection with im-axis :-

C.E. $s^4 + 8s^3 + 37s^2 + 50s + K = 0$

Routh array :-

s^4	1	37	K
s^3	8	50	0
s^2	30.75	K	0
s^1	$\frac{30.75 \times 50 - 8K}{30.75}$	0	0
s^0	K		

$$30.75 \times 50 - 8K = 0$$

$$\Rightarrow K = 192$$

So, $30.75 s^2 + 192 = 0$

$$\omega = \sqrt{\frac{192}{30.75}} = \sqrt{6}\ r/s.$$

Angle of departure :- $\phi_1 = -127^\circ$

$$\phi_2 = -105^\circ$$

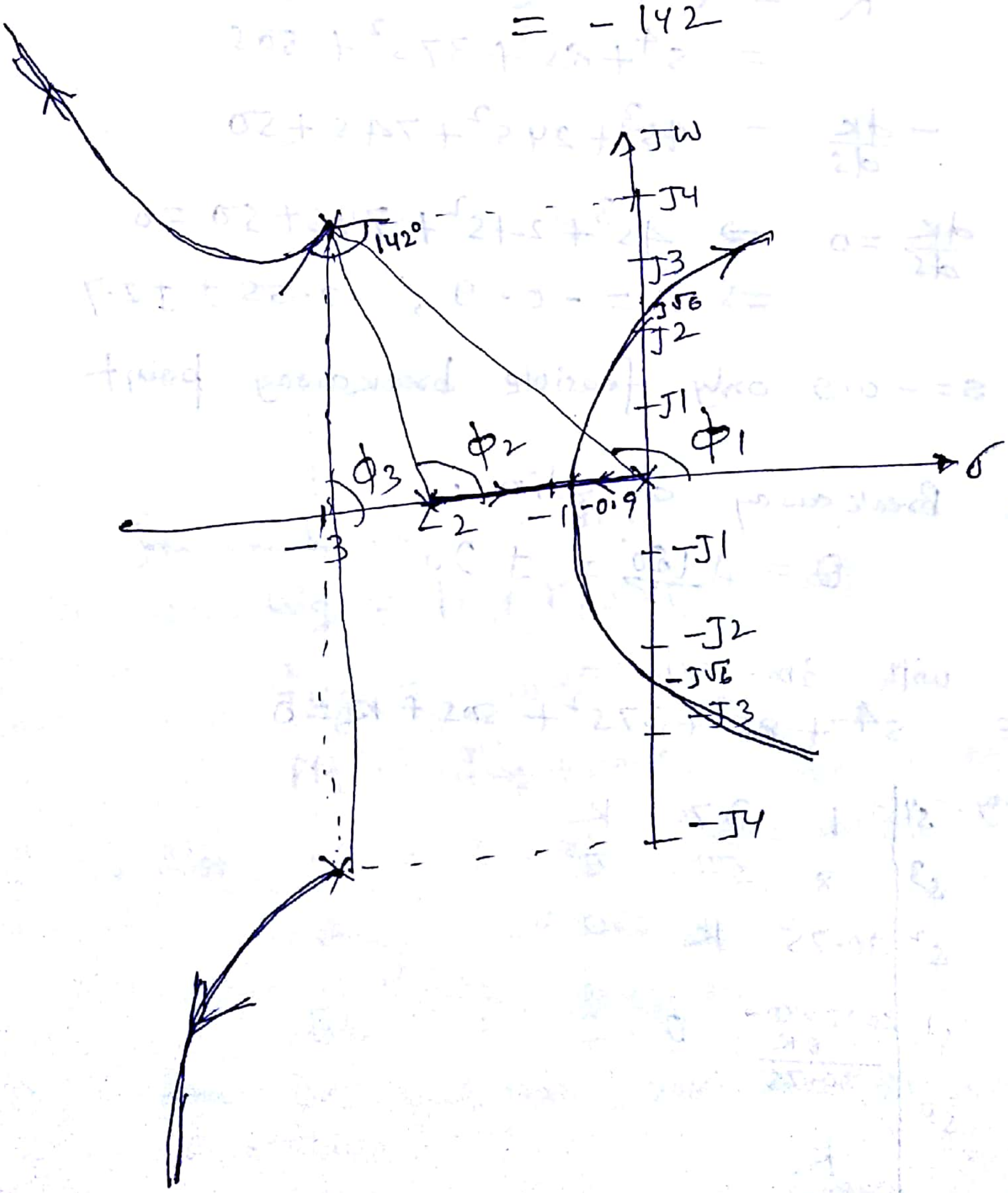
$$\phi_3 = -90^\circ$$

$$\Sigma\phi = -127^\circ - 105^\circ - 90^\circ$$

$$= -322^\circ$$

$$\phi_D = 180^\circ + \Sigma\phi = 180^\circ - 322^\circ$$

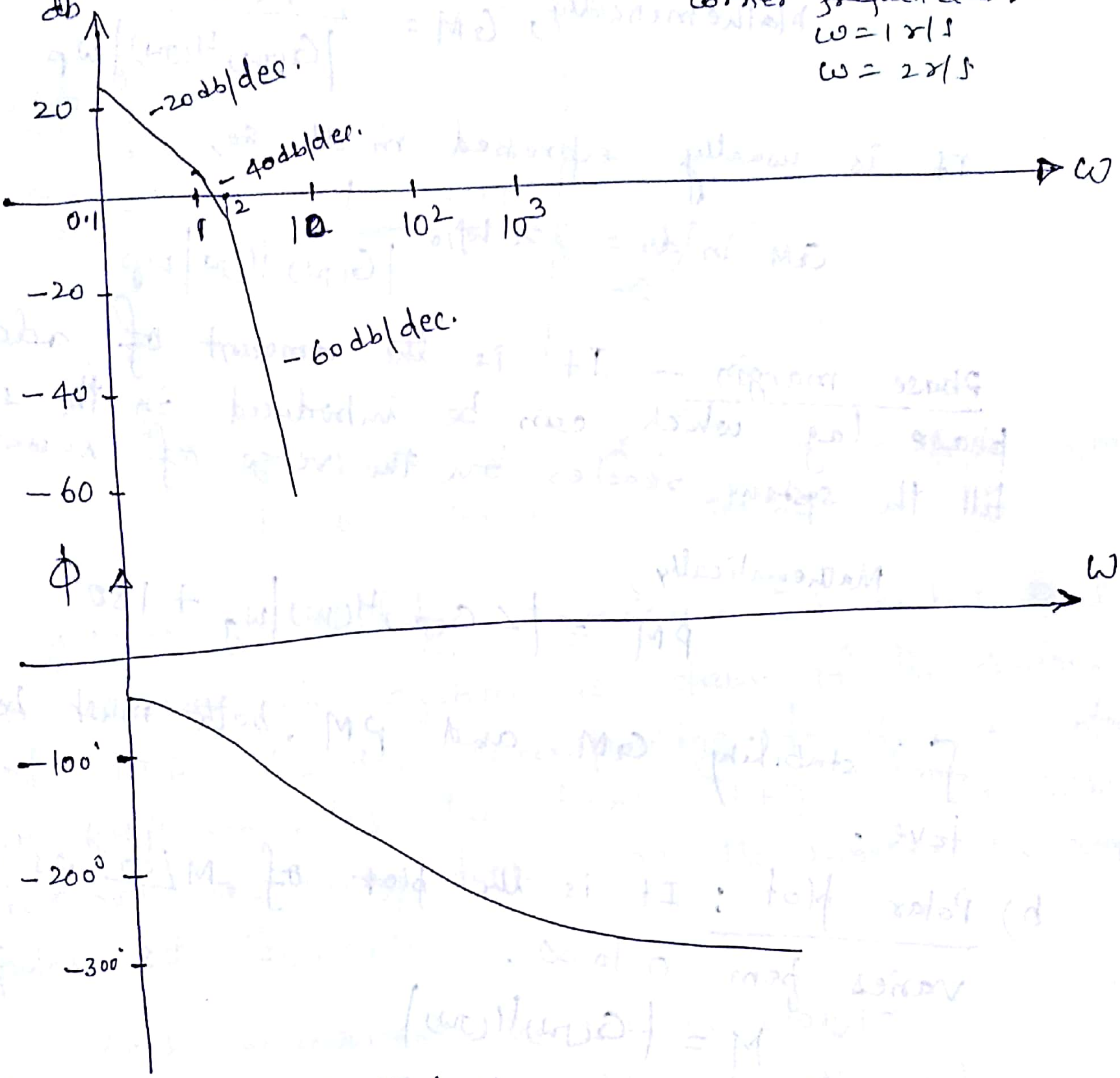
$$= -142^\circ$$



4. $G(j\omega) = \frac{4}{j\omega(1+j\omega)(2+j\omega)} = \frac{2}{j\omega(1+j\omega)(1+j0.5\omega)}$

Intercept of $20\log_{10}2 = 6$ db at $\omega = 1$ r/s.

Corner frequencies:
$\omega = 1$ r/s
$\omega = 2$ r/s



It can be observed that:

$\omega_g = 1.14$ r/s. (at 0 db)

$\phi_{\omega g} = -168°$

$PM = \phi_{\omega g} + 180° = -168° + 180° = 12°$

Also, $\omega_p = 1.4$ r/s (at $-180°$)

$db_{\omega_p} = -3.5$ db.

$GM = 0 - db_{\omega_p} = 0 - (-3.5) = 3.5$

Since, GM and PM both are positive so the system is stable.

(5) a) Gain margin – It is the factor by which the gain of the system can be increased till the system reaches on the verge of instability.

Mathematically, $GM = \dfrac{1}{|G(j\omega)H(j\omega)|_{\omega p}}$

It is usually expressed in db, so,

$$GM \text{ in db} = 20\log_{10}\frac{1}{|G(j\omega)H(j\omega)|_{\omega p}}$$

Phase margin – It is the amount of additional phase lag which can be introduced in the system till the system reaches on the verge of instability.

Mathematically,

$$PM = |\angle G(j\omega)H(j\omega)|_{\omega g} + 180^\circ$$

For stability GM and PM, both must be positive.

b) Polar plot : It is the plot of $M\angle\phi$ as frequency varies from 0 to ∞.

$$M = |G(j\omega)H(j\omega)|$$

and $\phi = \angle G(j\omega)H(j\omega)$

If polar plot does not enclose $-1+j0$ point then system is stable.
If polar plot encloses $-1+j0$ point, the system is unstable.

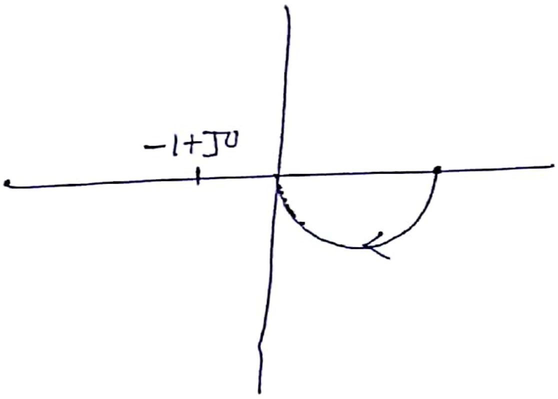
If polar plot passes through $-1+j0$ point then the system is marginally stable.

Ex. Polar plot of UFCS with OLTF

$$G(s) = \frac{10}{s+1}$$

$$G(j\omega) = \frac{10}{j\omega+1} = \frac{10}{\sqrt{\omega^2+1}} \angle -\tan^{-1}\omega$$

ω	$M\angle\phi$
0	$10\angle 0^\circ$
1	$7\angle -45^\circ$
⋮	
∞	$0\angle -90^\circ$



As polar plot does not enclose $-1+j0$ point, so the system is stable.

c) <u>Nyquist stability criterion</u> – It states that a closed-loop system is stable if the contour of OLTF $G(s)H(s)$ corresponding to Nyquist contour in RHP encircles the point $-1+j0$ in the CCW direction as many times as the number of RHP poles of $G(s)H(s)$.

Each encirclement in CCW direction, $N = 1$

" " CW " , $N = -1$

If $N = P$ then system is stable and if $N \neq P$, then system is unstable.

The no. of closed-loop poles lying in RHP (responsible for closed-loop instability)

$$Z = P - N.$$