B.Tech. - 2nd semester (Mid) - (2018 batch) Exam. 2019. (C+M+LT+IT).

Model-Answer.

Q.1. Ans:- (i) $N_2^+ \rightarrow \sigma(1s^2)\sigma^*(1s^2)\sigma(2s^2)\sigma^*(2s^2)(\pi 2p_x^2)(\pi 2p_y)^2(\sigma 2p_z)^1(\pi^* 2p_x^0 = \pi^* 2p_y^0)$

(ii) It enhance the conjugation in the molecule and enhancing the colour.

(iii) Heisenberg's Uncertainty Principle.

(iv) Dissociation energy is directly proportional to bond order.

(v) $Ca(HCO_3)_2 \xrightarrow{\Delta} CaCO_3\downarrow + H_2O + CO_2\uparrow$

$Mg(HCO_3)_2 \xrightarrow{\Delta} Mg(OH)\downarrow + 2CO_2\uparrow$

(vi) The basic Principle of NMR-spectroscopy describe the nuclei with spin quantum number (I) greater than zero can exhibit the NMR-phenomenon, when keeping the nuclei in a magnetic field and its interaction with radiofrequency.

(vii) The slow fluorescence is called phosphorescence.

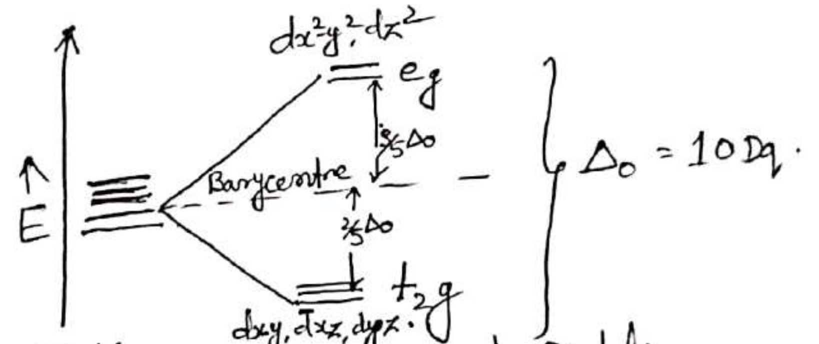
(viii) The kinetic energy of gaseous molecules is directly proportional to the absolute temperature.

(ix) BF_3

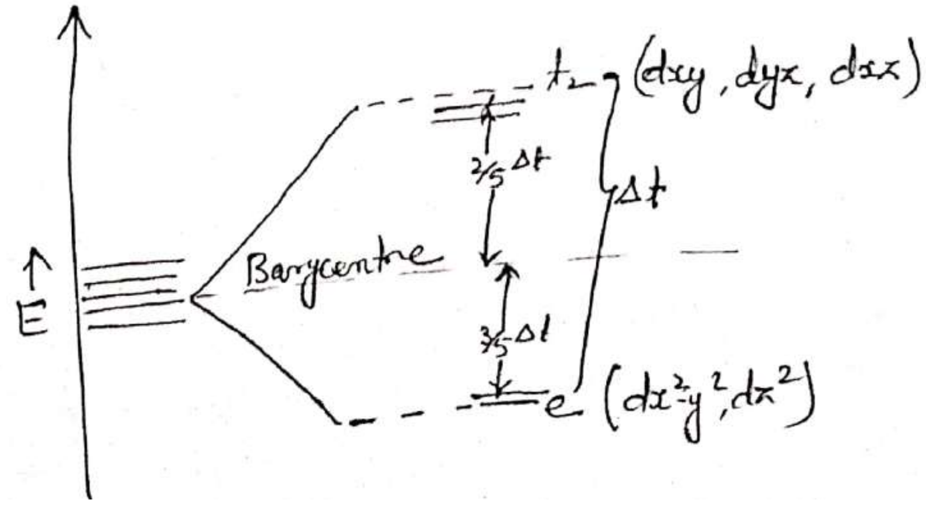
Q.2. (a) CFT:-
This theory replace valence bond theory for interpreting the chemistry of coordination compounds. "It was a model based on a purely electrostatic interaction between the ligands and the metal ion".
In the 1950s chemists begin to apply crystal field theory to transition metal complexes. The pure crystal field theory explain only the interaction between the metal ion and the ligands is an electrostatic or ionic one with the ligands being regarded as negative point charges. This theory is quite successful in interpreting many important properties of complexes. The symmetry considerations involved in the crystal-field approach are identical

to those of the molecular orbital Theory.
For crystal or ligand field effects in transition metal complexes, it describe the geometrical relationship of the d-orbitals. There is no unique way to representing the five d-orbitals, but, it is most convenient to explain the splitting of d-orbital in CFT as,

(i) Crystal field splitting in octahedral field:-
The splitting of d-orbitals in two sets, that is, as eg-set has two orbital of higher energy and t_2g-sets has three orbitals of lower energy. The total energy diffence of $\Delta_o = 10Dq$.

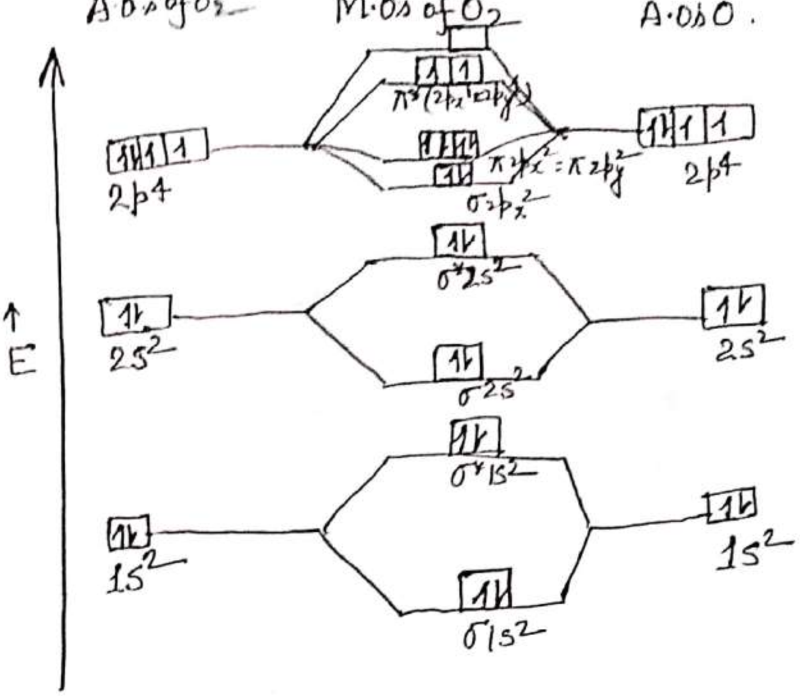


(ii) Crystal field splitting in tetrahedral field:-
The splitting of d-orbitals in CFT, the splitting of d-orbital also takes place in two parts as t_2-sets which have higher energy have three d-orbitals (dxy, dyz, dxz) and whereas e-sets which have two d-orbitals (dx^2y^2, dz^2) have lower energy



Q.2.(b) M.O. diagram of O_2:-

O_2 :- $(\sigma 1s)^2 (\sigma^* 1s)^2 (\sigma 2s)^2 (\sigma^* 2s)^2 (\sigma 2p_z)^2 (\pi 2p_x)^2 = (\pi 2p_y)^2, (\pi^* 2p_x)^1 (\pi^* 2p_y)^1 (\sigma^* 2p_z)^0$.

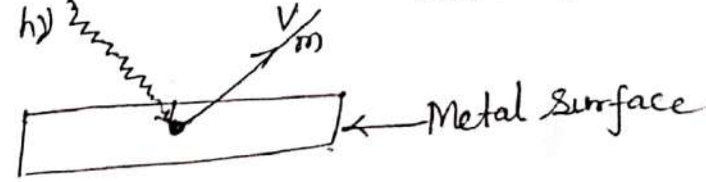


$B.O. = \frac{1}{2}(N_b - N_a) = \frac{1}{2}(10-6) = 4/2 = 2.$

Magnetic properties = Paramagnetic

Q.3. (a) Photoelectric effect :-

It is the phenomenon of ejection of electrons from a metal plate when light of a suitable wave-length falls on it. The electrons emitted are called Photoelectrons.



Results of photoelectric effect :-

(i) The electrons are ejected from the metal surface as soon as the beam of light strikes the surface.

(ii) The number of electrons ejected is directly proportional to the intensity or brightness of light.

(iii) Each metal has characteristic minimum frequency (ν_0) below which photoelectric effect is not observed. This minimum frequency is known as threshold frequency.

(iv) The K.E. of these electrons increases with the increase of frequency of the light used.

2.(b). Sol: Power of the bulb = 100 watt = $100 J s^{-1}$

Energy of one photon $E = h\nu = hc/\lambda$

$$= \frac{6.626 \times 10^{-34} Js \times 3 \times 10^{8} ms^{-1}}{400 \times 10^{-9} m}$$

$$= 4.969 \times 10^{-19} J$$

No. of Photons emitted $= \frac{100 Js^{-1}}{4.969 \times 10^{-19} J} = 2.012 \times 10^{20} s^{-1}$

Ans

Q.4 (a) Ans: Electronic Transitions/Excitations:-

The electronic excitations in a molecule can be broadly classified into;

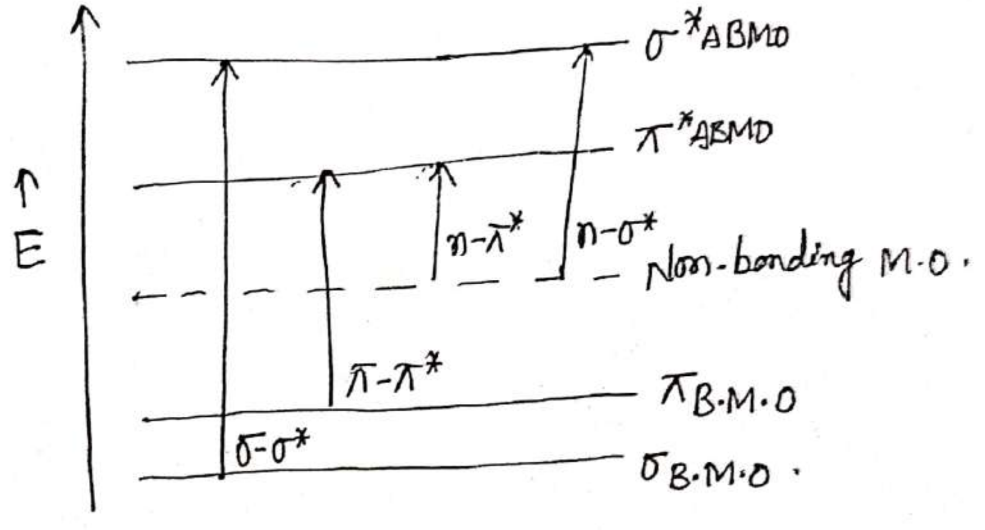
(i) $\sigma-\sigma^*$ transition:- As σ-electrons are held more firmly in the molecule, this transition takes place in UV or far UV-region.

(ii) $\pi-\pi^*$ transition:- This transition takes place in the near UV and Visible regions

(iii) $n-\pi^*$ transition:- This transitions are generally of weak intensities and lie in the visible region.

(iv) $n-\sigma^*$-transition:- In this transitions the required relatively less energy than $\sigma-\sigma^*$ transition.

$$\sigma \rightarrow \sigma^* > n \rightarrow \sigma^* > \pi \rightarrow \pi^* > n \rightarrow \pi^*$$



Q.4 (b). Sol:-

$$B = \frac{h}{8\pi^2 I c} = \frac{h}{8\pi^2 c} \times \frac{1}{I} = \frac{h}{8\pi^2 c} \times \frac{1}{\mu r^2}$$

$$B = \frac{h}{8\pi^2 c r^2} \times \frac{1}{\mu} \quad \text{—— (i)} \qquad \left(\because \mu = \frac{m_1 m_2}{m_1 + m_2}\right).$$

$$\Rightarrow 10.5909\ cm^{-1} = \frac{h}{8\pi^2 c r^2} \times \frac{m_1 + m_2}{m_1 \times m_2}$$

$$\Rightarrow 10.5909 = \frac{h}{8\pi^2 c r^2} \times \frac{(1+35)}{1 \times 35}$$

$$\Rightarrow 10.5909 = \frac{h}{8\pi^2 c r^2} \times \frac{36}{35}$$

$$\Rightarrow \frac{h}{8\pi^2 c r^2} = \frac{10.5909 \times 35}{36} = 10.2967\ cm^{-1}$$

Now. for $H^{37}Cl \rightarrow$ $B = \frac{h}{8\pi^2 c r^2} \times \frac{1}{\mu} = \frac{h}{8\pi^2 c r^2} \times \left(\frac{m_1 + m_2}{m_1 \cdot m_2}\right)$

$$= 10.2967 \times \frac{38}{37} = 10.5749\ cm^{-1}$$ — Ans

Similarly, for $^2D^{35}Cl \rightarrow$ $B = 10.296 \times \frac{37}{2 \times 35} = 5.4425\ cm^{-1}$ — Ans.

Q.5. (a) <u>Mechanism of SN^1 and SN^2 reactions:-</u>

<u>SN^1-mechanism:-</u> It is substitution nucleophilic unimolecular reaction.

Rate $\propto$ [substrate].

That is, When tert. butyl bromide reacts with potassium hydroxide, it give tert. butyl alcohol.

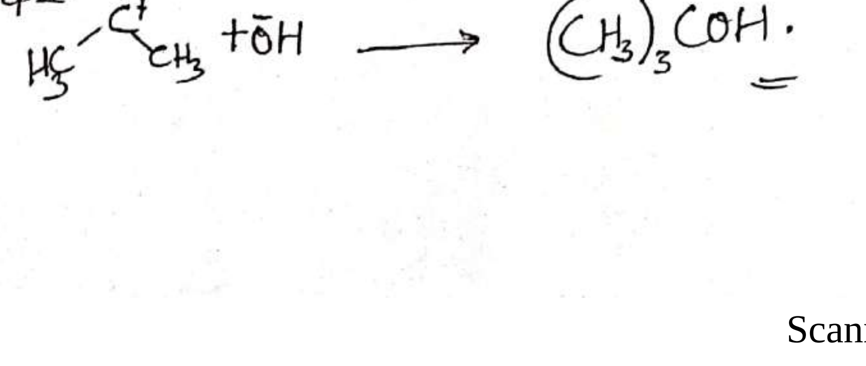
$$\underset{\text{tert. butyl bromide}}{(CH_3)_3C\text{-}Br} + \underset{\text{Potassium hydroxide}}{KOH} \longrightarrow \underset{\text{tert. butyl alcohol}}{(CH_3)_3COH} + KBr.$$

<u>Mechanism:-</u>

$$KOH \rightleftharpoons K^+ + \bar{O}H$$

Step I.

Step II.



Q. 5. (a) <u>SN_2 - reaction :-</u>

It is substitution nucleophilic bimolecular reaction. That is,

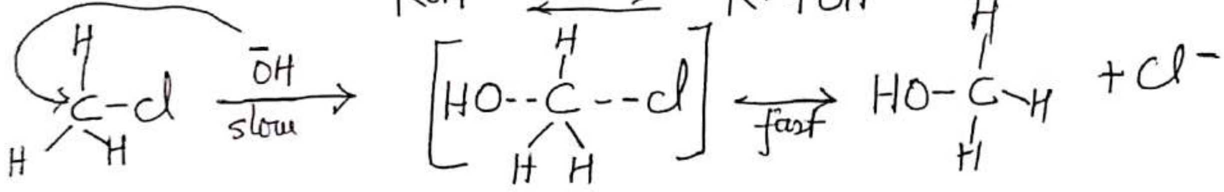
Rate $\propto$ [Substrate] [Nu].

e.g:- When methyl chloride reacts with alc. KOH, it gives methanol.

$$CH_3-Cl + alc.KOH \longrightarrow CH_3-OH + KCl.$$

Mechanism :-

$$KOH \rightleftharpoons K^+ + \bar{O}H$$



$$K^+ + Cl^- \longrightarrow KCl.$$

Q. 5. (a) <u>OR</u> :- <u>Addition Reaction :-</u>

The reaction in which two or more reactants reacts to give a product, such reaction is known as addition reaction. e.g:-

When propene reacts with HI, it gives 2-Iodopropane as the measure product via- Markovnikov's addition.

$$CH_3CH{=}CH_2 + HI \longrightarrow \underset{\substack{\text{1-Iodopropane}\\ \text{(minor)}}}{CH_3-CH_2-CH_2I} + \underset{\substack{\text{2-Iodopropane}\\ \text{(major)}}}{CH_3-CHI-CH_3}$$

<u>Elimination reaction :-</u>

The reaction in which two or more species eliminated from substrate molecule to give product, such reaction is known as elimination reaction. It is the reverse of addition reaction. e.g: of β-elimination reaction, when 2-Bromopentane is heated with alcoholic KOH solution, it gives Pent-2-ene as the major product according to Saytzeff rule.

$$H_3C-CH_2-CH_2-\overset{Br}{\overset{|}{C}}H-\underset{H}{CH_2} \longrightarrow \begin{cases} \xrightarrow{\bar{O}H} H_3C-CH_2-CH_2-CH{=}CH_2 \quad \text{Pent-1-ene (19\%)}. \\ \xrightarrow{\bar{O}H} H_3C-CH_2-CH{=}CH-CH_3 \quad \text{Pent-2-ene (81\%)}. \end{cases}$$

Q.5.(b) (i) $CH_3-CH=CH-CHO \xrightarrow[(ii)\,Zn,\,H_2O]{(i)\,O_3} CH_3-CHO + CH_3-CHO$

Ethanal.

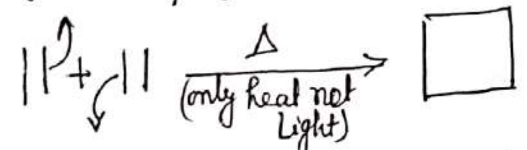
(ii) $\begin{matrix} CH-CH \\ \| \quad \| \\ CH-CH \end{matrix} \xrightarrow[(ii)\,Zn,\,H_2O]{(i)\,O_3} 2\ OHC-CHO$

(iii) $CH_3-CH_2-\overset{Br}{CH}-CH_3 \xrightarrow{alc.\,KOH} CH_3-CH=CH-CH_3 + CH_3-CH_2-CH=CH_2$

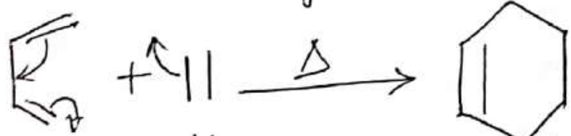
Major Minor

(iv) $CH_3-\overset{CH_3}{C}=CH_2 \xrightarrow{LiAlH_4} CH_3-\overset{CH_3}{\underset{H}{C}}-CH_3$

Q.6. (a) Cycloaddition reaction:- Combination of alkenes or polyenes to form a cyclic product. (Not require activation by light/No intermediate formation/cycloaddition no intermediate formation).



(2+2)



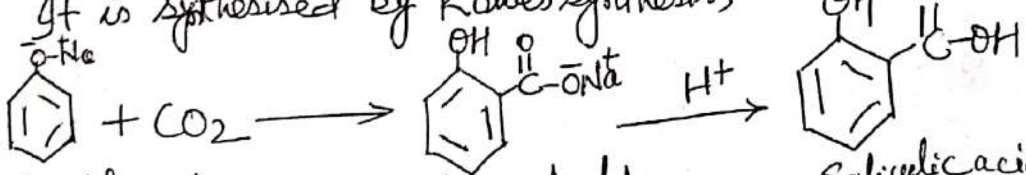
[4+2] cycloaddition

Diene Alkene (dienophile) Cyclohexene adduct

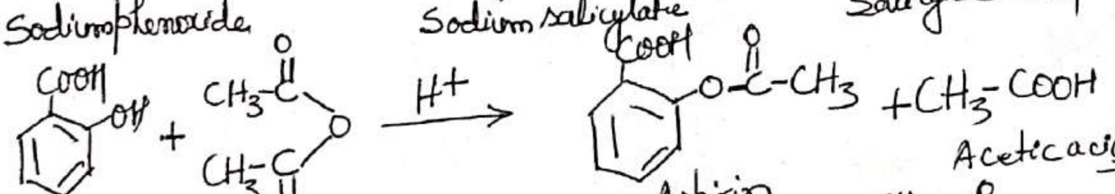
Dienophile — Loves a diene

(b) Mechanism of formation of Aspirin from Salicylic acid:-

It is sythesised by Kolbes synthesis;

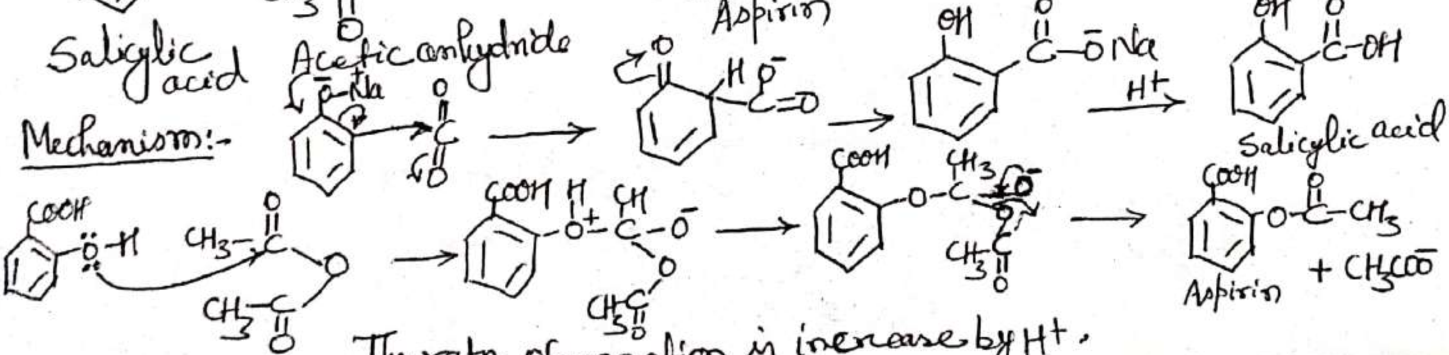


Sodium phenoxide Sodium salicylate Salicylic acid



Salicylic acid Acetic anhydride Aspirin Acetic acid

Mechanism:-



The rate of reaction is increase by H^+.

Q.7.(a) <u>Van der Waals Equation</u>/Equation of state for real gases:-

Let us consider an ideal gas equation,

$$PV = nRT \quad \text{—— (i)}$$

From postulates of kinetic molecular theory of gases, the volume and pressure corrections are required as,

<u>Volume correction:-</u>

The Volume 'V' of the gas is higher than the actual volume of the gas.

So, actual volume of the gas $= V - nb$ —— (ii)

where 'b' is van der Waals constant.

<u>Pressure Correction:-</u>

The pressure 'p' which is consider to the pressure of the gas is not the actual pressure of the gas but is smaller than the actual pressure.

So, actual pressure $P = P + p$

where 'p' is a measure of the attractive forces exerted by the number of molecules per unit volume.

$p \propto$ No. of molecules per unit volume

It has been observed that,

$p \propto d^2$ where 'd' = density of the gas

$p \propto \left(\frac{nm}{V}\right)^2$ where m is a constant

or $p \propto \frac{n^2}{V^2}$

or, $p = \frac{n^2 a}{V^2}$ where a = Constant known as 'van der Waals Constant'

Actual pressure of the gas $= \left(P + \frac{n^2 a}{V^2}\right)$ —— (iii)

putting the value of V and P from eq^n (ii) and (iii) in eq^n (i), we have,

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \quad \text{—— (iv)}$$

where a and b are Van der Waals constant

This is required equation of state for real gas.

Q.7. (b) (i) For an ideal gas $PV = nRT$, so, that

$$P = \frac{nRT}{V} = \frac{(1\ mol) \times (0.08206\ dm^3\ atm\ K^{-1}\ mol^{-1})(321\ K)}{1.32\ dm^3}$$

$$= 19.9\ atm.$$

Ans.

(ii) From a van der Waals gas equation,

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT$$

$$\Rightarrow P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}$$

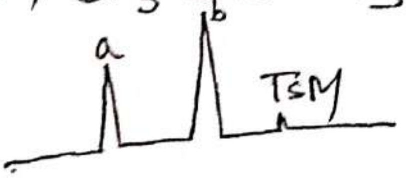
$$= \frac{(1\ mol)(0.08206\ dm^3\ atm\ K^{-1}\ mol^{-1})(321\ K)}{(1.32\ dm^3) - (1\ mol)(0.04267\ dm^3\ mol^{-1})} - \frac{(1\ mol)^2 (3.59\ dm^6\ atm\ mol^{-2})}{(1.32\ dm^3)^2}$$

[redacted] - $= 20.62\ atm - 2.06\ atm = 18.56\ atm.$ Ans.

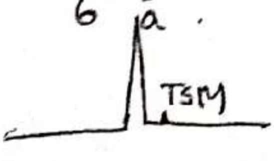
Q.8. (I) $\overset{c}{CH_3}-\overset{b}{CH_2}-O\overset{a}{H}$ → No. of NMR-signals 3

(ii) $\overset{b}{CH_3}-\overset{a}{CH_2}-Cl$ → 2

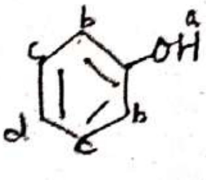
(iii) $\overset{b}{CH_3}-\overset{a}{CH}Cl-\overset{b}{CH_3}$ → 2



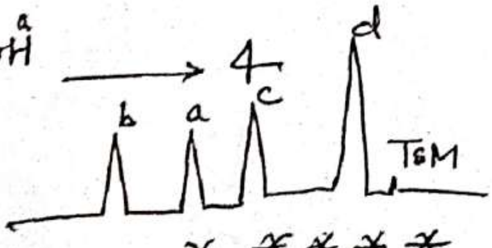
(iv) C_6H_6 → 1



(v) C_6H_5-OH →



→ 4



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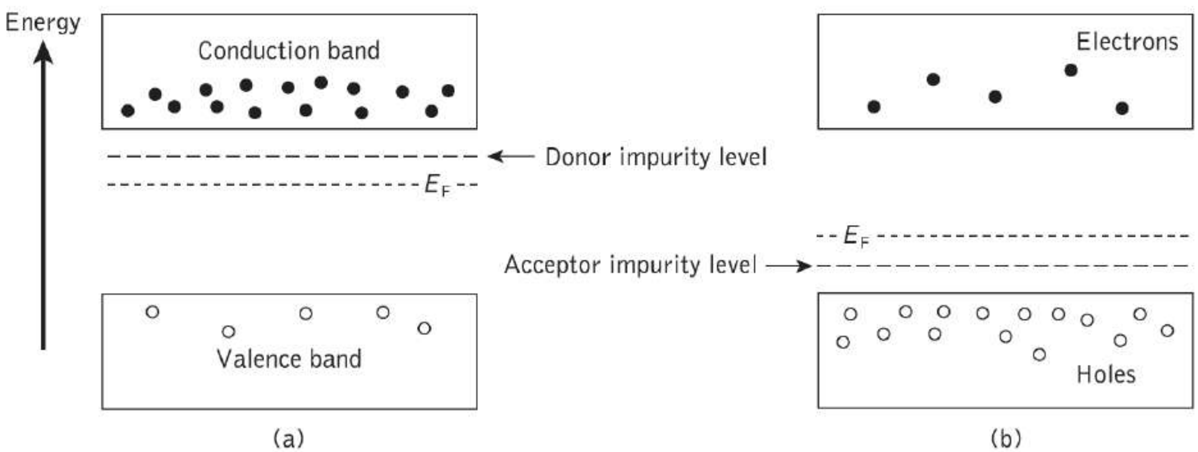
.**Question 1**

(A) **i**
(B) iv
(C) **i, iv**
(D) **Ii, iii**
(E) **iv**
(F) **ii**
(G) **ii**
(H) **iii**

.**Question 2:**

For a semiconductor in thermal equilibrium the energy-level occupation is described by the Fermi–Dirac distribution function (rather than the Boltzmann). Consequently, the probability $P(E)$ that an electron gains sufficient thermal energy at an absolute temperature T, such that it will be found occupying a particular energy level E, is given by the Fermi–Dirac distribution [Ref. 1]:



$$P(E) = \frac{1}{1 + \exp(E - E_F)/KT}$$

Question 3:

As the density of atoms in the lower or ground energy state E_1 is N_1, the rate of upward transition or absorption is proportional to both N_1 and the spectral density ρ_f of the radiation energy at the transition frequency f. Hence, the upward transition rate R_{12} (indicating an electron transition from level 1 to level 2) may be written as:

$$R_{12} = N_1 \rho_f B_{12}$$

where the constant of proportionality B_{12} is known as the Einstein coefficient of absorption.

The rate of stimulated downward transition of an electron from level 2 to level 1 may be obtained in a similar manner to the rate of stimulated upward transition. Hence the rate of stimulated emission is given by $N_2 \rho_f B_{21}$, where B_{21} is the Einstein coefficient of stimulated emission. The total transition rate from level 2 to level 1, R_{21}, is the sum of the spontaneous and stimulated contributions. Hence:

$$R_{21} = N_2 A_{21} + N_2 \rho_f B_{21}$$

For a system in thermal equilibrium, the upward and downward transition rates must be equal and therefore $R_{12} = \mathrm{R}_{21}$, or:

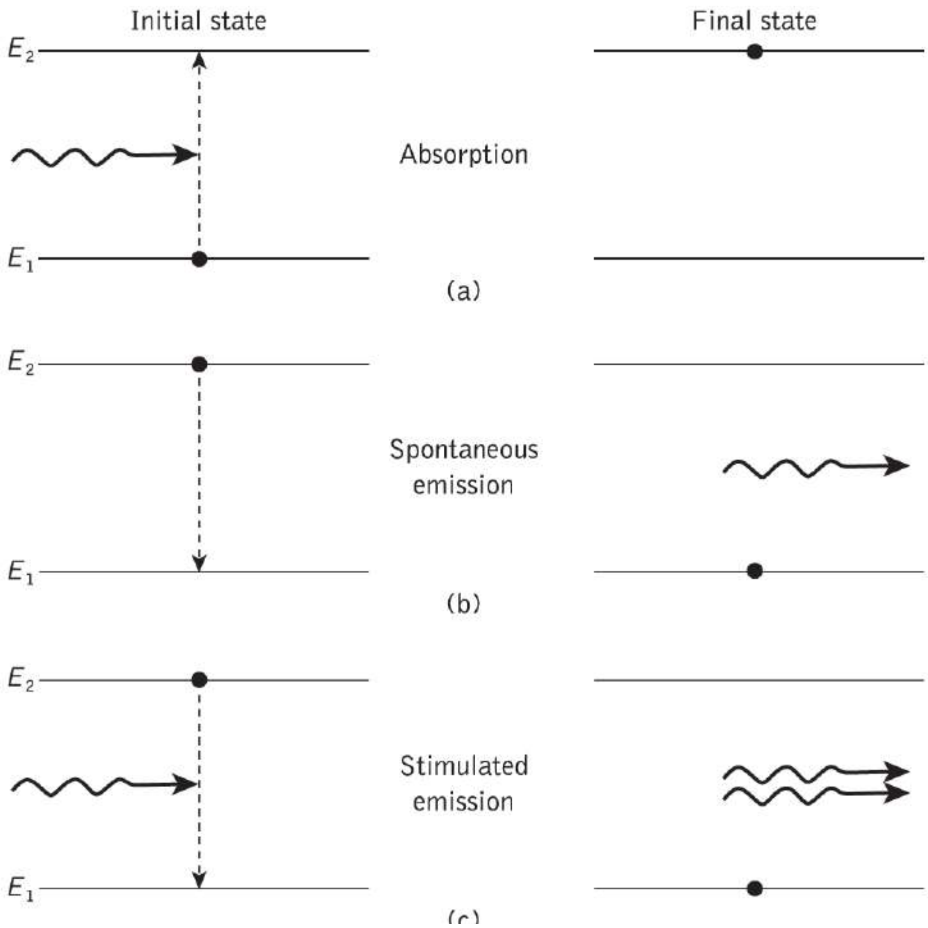
$$N_1 \rho_f B_{12} = N_2 A_{21} + N_2 \rho_f B_{21}$$

From this relation, we can derive the relation between Einstein's coefficient.

$$B_{12} = \left(\frac{g_2}{g_1} \right) B_{21}$$

and:

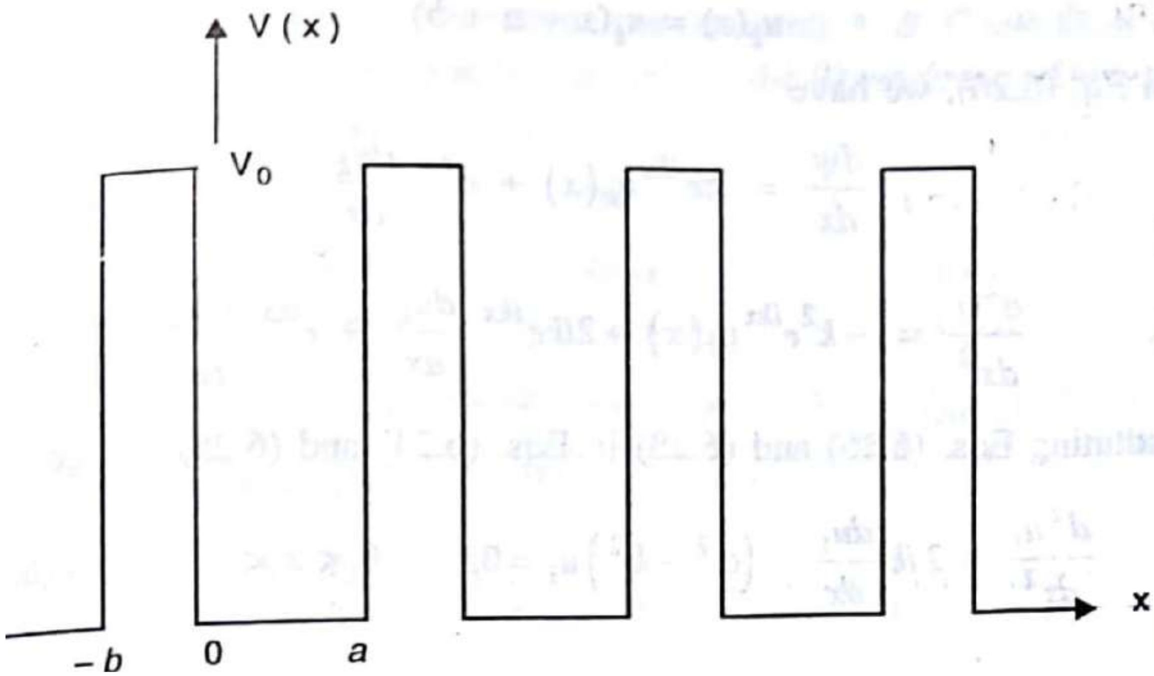
$$\frac{A_{21}}{B_{21}} = \frac{8 \pi h f^3}{c^3}$$



Question 4:

Consider the following idealized crystal potential:

We assume $E < V_o$



are obtained by writing the Schrodinger equations for the two regions as

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi = 0 \qquad \text{for } 0 < x < a \tag{6.21}$$

and

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} (E - V_o)\psi = 0 \qquad \text{for } -b < x < 0 \tag{6.22}$$

Assuming that the energy E of the electrons is less than V_o, we define two real quantities α and β as

$$\alpha^2 = \frac{2mE}{\hbar^2} \quad \text{and} \quad \beta^2 = \frac{2m(V_o - E)}{\hbar^2} \tag{6.23}$$

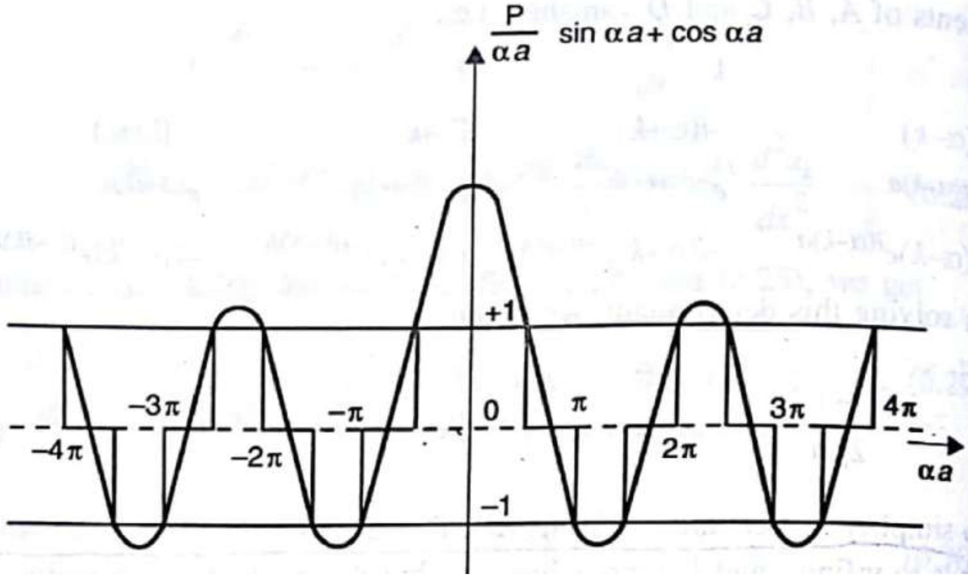
We define a quantity P as

because this

$$P = \frac{mV_0 ba}{\hbar^2} \tag{6.38}$$

which is a measure of the area $V_0 b$ of the potential barrier. Thus increasing P has the physical meaning of binding an electron more strongly to a particular potential well. Using Eq. (6.38) in (6.37), we get

$$P\frac{\sin \alpha a}{\alpha a} + \cos \alpha a = \cos ka \tag{6.39}$$



(i) The energy spectrum of the electrons consists of alternate regions of allowed energy bands (solid lines on abscissa) and forbidden energy bands (broken lines).

(ii) The width of the allowed energy bands increases with αa or the energy.

Question 8(iii): The governing equation that defines motion in 1D BOX may be written as,

$$\frac{d^2\psi_n}{dx^2} + \frac{2m}{\hbar^2} E_n\psi_n = 0$$

where E_n represents the kinetic energy of the electron in the nth state and V

is its potential energy.

The general solution to this equation is

$$\psi_n(x) = A \sin kx + B \cos kx$$

where

$$k = \sqrt{\frac{2mE_n}{\hbar^2}}$$

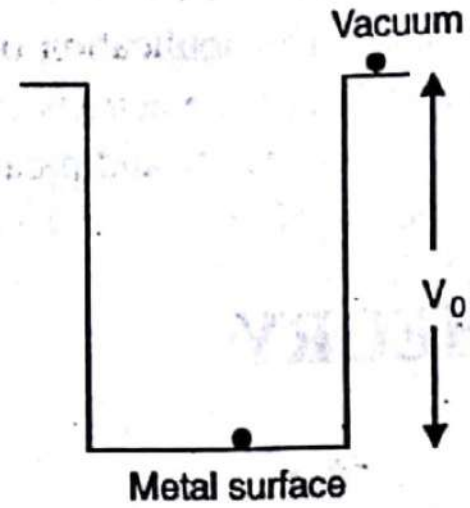
(i) Fermi Energy

If N is the total number of electrons to be accommodated on the line then, for even n, we can write

$$2n_F = N$$

where n_F represents the principal quantum number of the Fermi level. Thus, for $n = n_F$,

$$E_F = \frac{\hbar^2}{2m}\left(\frac{n_F\pi}{L}\right)^2 = \frac{\hbar^2}{2m}\left(\frac{N\pi}{2L}\right)^2$$



Density of States

The density of states is defined as the number of electronic states present in a unit energy range. It is denoted by $D(E)$ and is given by

$$D(E) = 2\frac{dn}{dE}$$

Hence

$$D(E) = \left(\frac{8mL^2}{h^2E}\right)^{1/2} = \frac{4L}{h}\left(\frac{m}{2E}\right)^{1/2}$$

Question 5:

The Harmonic Oscillator :-

For linear harmonic Oscillator,

$$V = \frac{kx^2}{2}, \quad k \rightarrow \text{Spring constant}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$2\pi\nu = \sqrt{\frac{k}{m}}$$

$$\omega = \sqrt{\frac{k}{m}}, \quad \frac{k}{m} = \omega^2$$

$$k = m\omega^2$$

$$V = \frac{1}{2}m\omega^2 x^2$$

Algebraic Method :-

The time independent SWE is.

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 = E\psi. \quad \text{—— (1)}$$

Rewrite above eqn as follows:

$$\frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right]\psi = E\psi. \quad \text{—— (2)}$$

Since, $(u^2+v^2 = (u+iv)(u-iv))$

Here, we can write (by assuming the defn)

$$a_\pm \equiv \frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d}{dx} \pm im\omega x\right)$$

Now,

$$(a_-a_+)f(x) = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right)\left(\frac{\hbar}{i}\frac{d}{dx} + im\omega x\right)f(x)$$

$$(a_-a_+)f(x) = \frac{1}{2m}\left(\frac{\hbar}{i}\frac{d}{dx} - im\omega x\right)\left(\frac{\hbar}{i}\frac{df}{dx} + im\omega xf\right)$$

$$= \frac{1}{2m}\left[-\hbar^2\frac{d^2f}{dx^2} + \hbar m\omega\frac{d}{dx}(xf) - \hbar m\omega x\frac{df}{dx} + (m\omega x)^2 f\right]$$

$$= \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 f + (m\omega x)^2 f + \hbar m\omega\left[x\frac{df}{dx} + f\right] - \hbar m\omega x\frac{df}{dx}\right]$$

$$= \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 f + (m\omega x)^2 + \hbar m\omega \cancel{x}\frac{df}{dx} + \hbar m\omega f - \hbar m\cancel{\omega}x\frac{df}{dx}\right]$$

$$= \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2 + \hbar m\omega\right]f$$

$$a_-a_+ f(x) = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2 + \hbar m\omega\right]f$$

Discarding the fn f from both side, we get.

$$a_-a_+ = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right] + \frac{1}{2}\hbar\omega$$

Now eqn (2) can be written as,

$$\left[a_-a_+ - \frac{1}{2}\hbar\omega\right]\psi = E\psi \quad \text{—— (3)}$$

Notice that, ordering of the factor a_+ and a_- is important here. So, we can get,

$$a_+a_- = \frac{1}{2m}\left[\left(\frac{\hbar}{i}\frac{d}{dx}\right)^2 + (m\omega x)^2\right] - \frac{1}{2}\hbar\omega$$

Thus.

$$a_-a_+ - a_+a_- = \hbar\omega$$

and in this case SWE is written as,

$$\left(a_+a_- + \frac{1}{2}\hbar\omega\right)\psi = E\psi \quad \text{—— (4)}$$

Now, here comes the crucial step:

Now,

$$\left(a_+a_- + \frac{1}{2}\hbar\omega\right)a_+\psi = \left(a_+a_-a_+ + \frac{1}{2}\hbar\omega a_+\right)\psi$$

$$= a_+\left[a_-a_+ + \frac{1}{2}\hbar\omega\right]\psi$$

$$= a_+\left[\left(a_-a_+ - \frac{1}{2}\hbar\omega\right)\psi + \hbar\omega\psi\right]$$

$$= a_+\left[E\psi + \hbar\omega\psi\right] = a_+(E+\hbar\omega)\psi$$

$$= (E+\hbar\omega)(a_+\psi)$$

This tells that, if ψ satisfies the SE, then $a_+\psi$ satisfies eqn with energy $(E+\hbar\omega)$.

Similarly, $a_-\psi$ is a solution with energy $E-\hbar\omega$.

proof :-

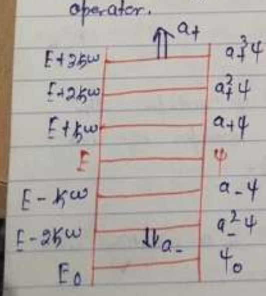
$$\left(a_-a_+ - \frac{1}{2}\hbar\omega\right)(a_-\psi) = a_-\left(a_+a_- - \frac{1}{2}\hbar\omega\right)\psi$$

$$= a_-\left[\left(a_+a_- + \frac{1}{2}\hbar\omega\right)\psi - \hbar\omega\psi\right]$$

$$= a_-\left[E\psi - \hbar\omega\psi\right]$$

$$= (E-\hbar\omega)(a_-\psi)$$

Here, $a_\pm$ are called ladder operator, a_+ is raising and a_- is lowering operator.



If I apply the lowering operator repeatedly then we reach with energy less than zero, which does not exist. Hence at some point (rung)

$$a_- \psi_0 = 0.$$

That is,

$$\frac{1}{\sqrt{2m}}\left(\frac{\hbar}{i}\frac{d\psi_0}{dx} - im\omega x\psi_0\right) = 0$$

or,

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar} x\psi_0$$

The above eqn for ψ_0 is easy to solve,

$$\int \frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}\int x\,dx$$

$$\ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 + \text{Constant}$$

So,

$$\boxed{\psi_0 = A_0 e^{-\frac{m\omega}{2\hbar}x^2}} \rightarrow \text{ground state}$$

To determine the energy of this state, we use (SWE) eqn (4).

$$\left[a_+a_- + \frac{1}{2}\hbar\omega\right]\psi_0 = E_0\psi_0$$

$$a_+(a_-\psi_0) + \frac{1}{2}\hbar\omega\psi_0 = E_0\psi_0$$

$$0 + \frac{1}{2}\hbar\omega\psi_0 = E_0\psi_0 \quad (\because a_-\psi_0 = 0)$$

$$\boxed{E_0 = \frac{1}{2}\hbar\omega} \rightarrow \text{ground state.}$$

Now, we simply apply a_+ to generate excited state.

$$\boxed{\begin{aligned}\psi_n(x) &= A_n (a_+)^n e^{-\frac{m\omega}{2\hbar}x^2}\\ \text{and } E_n &= \left(n+\frac{1}{2}\right)\hbar\omega\end{aligned}}$$

Question 6:

Particle in 3D-Box:

Let us consider a particle of mass m in a rectangular box of sides a, b, c.

$$V(x,y,z) = \begin{cases} 0, & 0<x<a,\ 0<y<b,\ 0<z<c \\ \infty, & \text{elsewhere} \end{cases}$$

or,

$$V(x,y,z) = V_x(x) + V_y(y) + V_z(z)$$

$$V_x(x) = \begin{cases} 0, & 0<x<a \\ \infty, & \text{elsewhere.} \end{cases}$$

The Schrodinger wave eqn,

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi,$$

Here

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(Laplacian)

$$\nabla^2\psi + \frac{2mE}{\hbar^2}\psi = 0 \quad \text{—— (1)}$$

The soln of the above eqn would be

$$\psi(x,y,z) = \psi_x\psi_y\psi_z$$

Substituting ψ in eqn (1), ~~and~~

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\psi_y\psi_z\frac{\partial^2\psi_x}{\partial x^2} + \psi_x\psi_z\frac{\partial^2\psi_y}{\partial y^2} + \psi_x\psi_y\frac{\partial^2\psi_z}{\partial z^2} + \frac{2mE}{\hbar^2}\psi_x\psi_y\psi_z = 0$$

Dividing above eqn by $\psi_x\psi_y\psi_z$, we get,

$$\frac{1}{\psi_x}\frac{\partial^2\psi_x}{\partial x^2} + \frac{1}{\psi_y}\frac{\partial^2\psi_y}{\partial y^2} + \frac{1}{\psi_z}\frac{\partial^2\psi_z}{\partial z^2} + \frac{2mE}{\hbar^2} = 0 \quad \text{—— (2)}$$

Now, $E = (E_x + E_y + E_z)$

or,

$$\frac{1}{\psi_x}\frac{\partial^2\psi_x}{\partial x^2} + \frac{1}{\psi_y}\frac{\partial^2\psi_y}{\partial y^2} + \frac{1}{\psi_z}\frac{\partial^2\psi_z}{\partial z^2} = -\frac{2m}{\hbar^2}(E_x + E_y + E_z)$$

Using the separation of variables technique,

$$\frac{1}{\psi_x}\frac{\partial^2\psi_x}{\partial x^2} = -\frac{2mE_x}{\hbar^2} \quad \text{—— (3)}$$

$$\frac{1}{\psi_y}\frac{\partial^2\psi_y}{\partial y^2} = -\frac{2mE_y}{\hbar^2} \quad \text{—— (4)}$$

$$\frac{1}{\psi_z}\frac{\partial^2\psi_z}{\partial z^2} = -\frac{2mE_z}{\hbar^2} \quad \text{—— (5)}$$

Solving eqn (3),

$$\frac{\partial^2\psi_x}{\partial x^2} + \frac{2mE_x}{\hbar^2}\psi_x = 0$$

$$\frac{\partial^2 \psi_x}{\partial x^2} + k_x^2 \psi_x = 0 \quad \text{------} (6)$$

where $k_x^2 = \frac{2mE_x}{\hbar^2}$

The solⁿ of the eqn (6) will be,

$$\psi_x = A \sin k_x x + B \cos k_x x$$

As solved previously,

$$\psi_x = \sqrt{\frac{2}{a}} \sin\left(\frac{n_x \pi}{a} x\right), \quad n_x = 1,2,3..$$

Similarly,

$$\psi_y = \sqrt{\frac{2}{b}} \sin\left(\frac{n_y \pi y}{b}\right), \quad n_y = 1,2,3...$$

and $\psi_z = \sqrt{\frac{2}{c}} \sin\left(\frac{n_z \pi}{c} z\right), \quad n_z = 1,2,3...$

Hence,

$$\boxed{\psi_{n_x n_y n_z}(x,y,z) = \sqrt{\frac{8}{abc}} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{b} y\right) \sin\left(\frac{n_z \pi}{c} z\right)}$$

$$\text{and} \quad \boxed{E_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)}$$

For Cubic Potential,

$$a = b = c = L$$

Then,

$$E = \frac{\hbar^2 \pi^2}{2mL^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

and

Question 7:

Problem :- Find the probability of a particle trapped in a box L betⁿ 0.45L and 0.55L for the ground state and first excited state.

Solⁿ :- We know that for particle in a box

$$\psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

The probability of finding the particle,

$$P = \int_{x_1}^{x_2} |\psi_n(x)|^2 dx$$

$$= \frac{2}{L} \int_{x_1}^{x_2} \sin^2 \frac{n\pi x}{L} dx$$

$$= \frac{2}{L} \int_{x_1}^{x_2} \left[\frac{1 - \cos \frac{2n\pi x}{L}}{2}\right] dx.$$

$$= \frac{1}{L}\left[x - \frac{\sin 2n\pi x / L}{\frac{2n\pi}{L}}\right]_{x_1}^{x_2}$$

$$= \left[\frac{x}{L} - \frac{1}{2n\pi} \sin \frac{2n\pi x}{L}\right]_{x_1}^{x_2}$$

For ground state, $n = 1$

$$P = \left[\frac{x}{L} - \frac{1}{2\pi} \sin \frac{2\pi x}{L}\right]_{x_1}^{x_2}$$

$$P = \left[\frac{x}{L} - \frac{1}{2\pi} \sin \frac{2\pi x}{L}\right]_{0.45L}^{0.55L}$$

For first excited state, $n = 2$

$$P = \left[\frac{x}{L} - \frac{1}{4\pi} \sin \frac{4\pi x}{L}\right]_{0.45L}^{0.55L}$$

Question 8: (I)

Probability Current $\Rightarrow$

We know that, the probability density,

$$P = \psi^* \psi$$

$$\frac{\partial P}{\partial t} = \left[\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi\right] \quad \text{—— (1)}$$

Now, the time-dependent Schrodinger wave eqn is:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \quad \text{— (2)}$$

and its complex conjugate i.e.

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + V(x)\psi^*$$

$(\because V^* = V)$ potential is always real)

$$\frac{\partial \psi^*}{\partial t} = \frac{\hbar^{\cancel{2}}}{2mi\cancel{\hbar}}\frac{\partial^2 \psi^*}{\partial x^2} - \frac{V(x)\psi^*}{i\hbar} \quad \text{—— (3)}$$

$$\frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{iV(x)\psi^*}{\hbar}$$

~~Module - 1~~

~~Solution of Wave eqn~~

From eqn (2),

$$\frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2mi\hbar}\frac{\partial^2 \psi}{\partial x^2} + \frac{V(x)\psi}{i\hbar}$$

$$\frac{\partial \psi}{\partial t} = \frac{-\hbar}{2mi}\frac{\partial^2 \psi}{\partial x^2} + \frac{V(x)\psi}{i\hbar}$$

$$\frac{\partial \psi}{\partial t} = \frac{-i\hbar}{2mi^2}\frac{\partial^2 \psi}{\partial x^2} + \frac{V(x)\psi}{i\hbar}$$

$$\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{iV(x)\psi}{\hbar} \quad \text{— (4)}$$

Similarly, from eqn (3),

$$\frac{\partial \psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{iV(x)\psi^*}{\hbar} \quad \text{— (5)}$$

Putting eqn (4) and (5) in eqn (1) we get,

$$\frac{\partial P}{\partial t} = \psi^*\left[\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{iV\psi}{\hbar}\right] + \left[-\frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{iV\psi^*}{\hbar}\right]\psi$$

$$\frac{\partial P}{\partial t} = \frac{i\hbar}{2m}\psi^*\frac{\partial^2 \psi}{\partial x^2} - \cancel{\frac{iV}{\hbar}\psi^*\psi} - \frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2}\cdot\psi + \cancel{\frac{iV}{\hbar}\psi^*\psi}$$

$$\frac{\partial P}{\partial t} = \frac{i\hbar}{2m}\left[\psi^*\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2}\psi\right]$$

$$\frac{\partial P}{\partial t} = \frac{i\hbar}{2m}\left[\psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}\right]$$

Now, we define,

$$\frac{\partial}{\partial x}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]$$

$$= \psi^*\frac{\partial^2 \psi}{\partial x^2} + \cancel{\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}} - \psi\frac{\partial^2 \psi^*}{\partial x^2} - \cancel{\frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x}}$$

$$= \left[\psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}\right]$$

Hence,

$$\frac{\partial P}{\partial t} = \frac{i\hbar}{2m}\frac{\partial}{\partial x}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]$$

we define,

$$\boxed{j(r,t) \equiv \frac{\hbar}{2m}\left[\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]}$$

probability current

$$\frac{\partial P}{\partial t} = -\frac{\partial j}{\partial x}$$

$$\boxed{\frac{\partial P}{\partial t} + \frac{\partial j}{\partial x} = 0} \rightarrow \text{Conservation of Probability.}$$

This is continuity eqn.

This means the probability does not depends upon time. It can be proved as

$$\frac{d}{dt}\int_{-\infty}^{\infty} dx \, |\psi(x,t)|^2$$

$$= \int_{-\infty}^{\infty} dx \, \frac{\partial}{\partial t}|\psi(x,t)|^2$$

$$= \int_{-\infty}^{\infty} dx \left(-\frac{\partial j}{\partial x}\right)$$

$$= -[j(\infty,t) - j(-\infty,t)]$$

$$= 0.$$

we assume that the wave f^n vanishes at infinity, i.e.

$$\boxed{\frac{d(P)}{dt} = 0.}$$

(ii)

Which is a [illegible]

Free particle wave f^n and wave packets :-

Wave packets :-

A localized wave f^n is called a wave packet. It consists of a group of waves of slightly differed wavelengths so chosen that they interfere constructively over a small region of space and destructively elsewhere. Mathematically, we can carry out this type of interference or superposition by means of Fourier transforms. We can construct the wave packet $\psi(x,t)$ by superposing the plane waves (propagating along the x-axis) of different frequencies.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk$$

$\phi(k)$ is the amplitude of the wave packet.

At $t=0$,

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{ikx}\, dk .$$

where $\phi(k)$ is the Fourier transform of $\psi_0(x)$.

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(x)\, e^{-ikx}\, dx .$$

Free particle wave f^n :-

This is the simplest one-dimensional problem because it corresponds to $V(x)=0$. In this case, SWE is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + 0 = E\psi(x)$$

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

$\Rightarrow$ $k' = k = \frac{p}{\hbar}$

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = 0$$

where $k^2 = \frac{2mE}{\hbar^2}$

The general sol^n of above eq^n is,

$$\underbrace{\psi(x)}_{\text{eigenf}^n} = A_+ e^{ikx} + A_- e^{-ikx}$$

where A_+ and A_- are arbitrary constants.

The complete wave f^n,

$$\underbrace{\psi(x,t)}_{\text{wave } f^n} = \underbrace{A_+ e^{i(kx-\omega t)}}_{\text{wave travelling to the right}} + \underbrace{A_- e^{-i(kx+\omega t)}}_{\text{wave travelling to the left}}$$

Since, there are no restrictions, hence p and E can take any values. It is a simple problem but presents a no. of physical subtleties. Let us discuss three of them:

(i) $P_\pm(x,t) = |\psi_\pm(x,t)|^2 = |A_\pm|^2$

are constant and does not depend on x ~~and~~.

(ii) The speed of plane wave is,

$$v_{wave} = \frac{\omega}{k} = \frac{E}{\hbar k} = \frac{\hbar^2 k^2/2m}{\hbar k}$$

$$= \frac{\hbar k}{2m}$$

$$v_{classical} = \frac{p}{m} = \frac{\hbar k}{m} = 2\, v_{wave} .$$

This means that the particle travels twice as fast as the wave that represents it.

(iii) The wave f^n is not normalizable:

$$\int_{-\infty}^{\infty} \psi_\pm^* \psi_\pm(x,t)\, dx = |A_\pm|^2 \int_{-\infty}^{\infty} dx$$

So, A must be zero because, $\int_{-\infty}^{\infty} dx = \infty$

This is not physical.

Thus, the sol^n $\psi_\pm(x,t)$ is unphysical. A free particle cannot have sharply defined momenta and energy. Thus, the sol^n in this cannot be plane waves but it should be wave packets:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk .$$

The wave packet sol^n cures and avoids all subtleties raised above.

In summary, a free particle cannot be represented by a single plane wave; but it has to be represented by a wave packet.

Muzaffarpur Institute of Technology
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2nd Semester (Mid-Semester) Exam – 2018-19

Civil, Information Technology, Mechanical and Leather Technology

ENGLISH (100106)

Answer Key

Q1.

a)

i) At.

ii) Along

b)

i) The

ii) A

c)

i) The needy people should be helped.

ii) You are requested to post this letter

d)

i) Both, On both sides.

ii) Book

iii) Speak/Declare

iv) Work

e)

i) **I had** my dinner an hour ago.

ii) She **slept** for eight hours last night.

f)

i) A

ii) C

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Ag

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Solution 2nd Semester

Mathematics II

ECE and EE

1. (a). (iii) Δ

(b). (i). $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(c). (iv) 2

(d). (iii) either zero or purely imaginary.

(e). —

Solution 2 (a)

② (a) $L\left\{ \int_0^t \frac{\sin u}{u}\, du \right\}$

Soln

$\because \quad L\{\sin u\} = \dfrac{1}{s^2+1}$

$L\left\{\dfrac{\sin u}{u}\right\} = \displaystyle\int_s^\infty \frac{1}{s^2+1}\, ds \qquad \left[\text{As } L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s)\,ds\right]$

$= \left[\tan^{-1} s\right]_s^\infty$

$= \dfrac{\pi}{2} - \tan^{-1} s$

$L\left\{\dfrac{\sin u}{u}\right\} = \cot^{-1} s. = f(s) \quad (\text{Say})$

~~Now As~~ $L\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \dfrac{f(s)}{s}$

$= \dfrac{\cot^{-1} s}{s}$

Now As $L\left\{\int_0^t f(t)\, dt\right\} = \dfrac{f(s)}{s}$

$\therefore \quad L\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \dfrac{\cot^{-1} s}{s}$

Solution 2 (b)

We know that $L(e^t \sin t) = \dfrac{1}{(s-1)^2+1}$

Hence
$$L\left(\frac{e^t \sin t}{t}\right) = \int_s^\infty \frac{ds}{(s-1)^2+1} = \left[\tan^{-1}(s-1)\right]_s^\infty$$
$$= \frac{\pi}{2} - \tan^{-1}(s-1) = \cot^{-1}(s-1)$$

Therefore
$$L\left[\int_0^t \frac{e^t \sin t}{t}\, dt\right] = \frac{1}{s}\cot^{-1}(s-1)$$ **Ans.**

Solution 3

SOLUTION: (*i*) Since $L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{1}{a}\sin at$, using convolution theorem here, we get

$$L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right\} = \int_0^t \frac{1}{a}\sin au \cdot \frac{1}{a}\sin a(t-u)\, du$$
$$= \frac{1}{2a^2}\int_0^t [\cos a(2u-t) - \cos at]\, du = \frac{1}{2a^2}\left[\frac{1}{2a}\sin a(2u-t) - u\cos at\right]_0^t$$
$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin at - t\cos at + \frac{1}{2a}\sin at\right] = \frac{1}{2a^3}[\sin at - at\cos at].$$ **Ans.**

(*ii*) Since $L\left(\dfrac{t}{2a}\sin at\right) = \dfrac{s}{(s^2+a^2)^2}$ and $L(\sin at) = \dfrac{a}{s^2+a^2}$, applying convolution theorem we get

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\frac{1}{(s^2+a^2)}\right] = \int_0^t \frac{u}{2a}\sin au \cdot \frac{1}{a}\sin a(t-u)\, du = \frac{1}{2a^2}\int_0^t u\sin au \sin a(t-u)\, du$$
$$= \frac{1}{4a^2}\int_0^t u[\cos(2au-at) - \cos at]\, du$$
$$= \frac{1}{4a^2}\int_0^t u\cos(2au-at)\, du - \frac{1}{4a^2}\left[\frac{u^2}{2}\cos at\right]_0^t$$

$$= \frac{1}{4a^2}\left[\left\{u\,\frac{\sin(2au-at)}{2a}\right\}_0^t - \int_0^t \frac{1\cdot\sin(2au-at)}{2a}\, du\right] - \frac{t^2}{8a^2}\cos at$$
$$= \frac{1}{4a^2}\left[\frac{t}{2a}\sin at + \frac{1}{4a^2}\{\cos(2au-at)\}_0^t\right] - \frac{t^2}{8a^2}\cos at$$
$$= \frac{t}{8a^3}\sin at + \frac{1}{16a^4}(\cos at - \cos at) - \frac{t^2}{8a^2}\cos at$$
$$= \frac{t}{8a^3}(\sin at - at\cos at).$$ **Ans.**

Solution 4 (a)

The matrix eqn is given as

$$\begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ \mu \end{bmatrix}$$

It will have unique solution if Coefficient matrix is of rank 3. This requires that $\begin{vmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & \lambda \end{vmatrix} = 15(5-\lambda) \neq 0$

Thus for unique solution $\lambda \neq 5$ and μ may have any value

If $\lambda = 5$, the system of eqn will have no solution for those values of μ for which the matrices

$A = \begin{bmatrix} 2 & 3 & 5 \\ 7 & 3 & -2 \\ 2 & 3 & 5 \end{bmatrix}$ and augmented matrix $K = \begin{bmatrix} 2 & 3 & 5 & 9 \\ 7 & 3 & -2 & 8 \\ 2 & 3 & 5 & \mu \end{bmatrix}$

are not of the same rank. But A is of rank 2 and K is not of rank 2, unless $\mu = 9$. Thus if $\lambda = 5$ and $\mu \neq 9$ the system will have no solution.

If $\lambda = 5$ and $\mu = 9$, the system will have infinite solution

Ans

Solution 4 (b)

(b) The characteristic eqn of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley-Hamilton theorem, we have to show that $A^3 - 6A^2 + 9A - 4I = 0$

$$\therefore A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

This verifies the theorem.

Now multiplying both sides by A^{-1}, we get

$$A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$ Ans

Solution 5 (a)

The characteristic eqn of A is given by

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)(\lambda-3) = 0$$

$\therefore \lambda = 1, 2, 3$ are distinct eigenvalues of A.

For $\lambda = 1$, the matrix eqn $[A - I]X = 0$ gives eigenvector

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} -x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ \text{or } x_1 + x_2 = 0 \end{matrix}$$

The Solution is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda = 2$, eigenvector is given by Solution of $[A - 2I]X = 0$

Thus $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ The Solution is $X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda = 3$, eigen vector is obtained from eqn $[A - 3I]X = 0$

Thus $\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The Solution is $X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

Hence the required eigenvectors are $\begin{bmatrix} 1 & -2 & -1 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$ Ans

Solution 5 (b)

The characteristic eqn of A is given by

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \quad \therefore \lambda = 1, 2, 3$$

Since matrix A has three distinct eigenvalues, it has three linearly independent eigenvectors hence it is diagonalizable.

The eigenvector corresponding to $\lambda = 1$, is given by

$$[A - I]X = 0 \Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

The eigenvector corresponding to $\lambda = 2$ is given by

$$[A - 2I]X = 0 \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The eigenvector corresponding to $\lambda = 3$ is given by

$$[A - 3I]X = 0 \Rightarrow \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution is $X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Hence the modal matrix $P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

and $P^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

$\therefore P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ which is a diagonal matrix

Solution 6 (a)

Symmetric Matrix: A symmetric matrix is a square matrix that is equal to its transpose.

Ex. $\begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 4 \\ 1 & 4 & 6 \end{bmatrix}$

Orthogonal Matrix: A square matrix A is said to be orthogonal if $AA^T = I$.

Ex. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Hermitian Matrix: A square matrix A is said to be Hermitian if A=A*(transposed conjugate of A).

Ex. $\begin{bmatrix} 2 & -3+i & 1+2i \\ -3-i & 5 & 4-i \\ 1-2i & 4+i & 6 \end{bmatrix}$

Solution 6 (b)

We know that λ is said to be an eigenvalue of a square matrix A if there is a nonzero column vector X s. t.

$$AX=\lambda X \text{..............................} (1)$$

More over the system of equations AX=0 has non trivial solutions iff |A|=0.
Now from (1), we have $\lambda=0 \Leftrightarrow AX=0 \Leftrightarrow |A|=0$ (since $X \neq 0$).
This proves that A is singular iff $\lambda=0$.

Solution 7 (a)

Sol^n:- Here $f(x,y) = x^3 + y/2$
and $h = 0.1$, $x_0 = 1$, $y_0 = 2$, $x_1 = 1.1$, $x_2 = 1.2$

For finding y_1:-

$k_1 = hf(x_0, y_0) = 0.1 f(1,2)$
$= 0.2$
$k_2 = 0.1 f(1 + .1/2,\ 2 + .2/2)$
$= 0.22076$
$k_3 = 0.1 f(1 + .1/2,\ .22076/2 + 2)$
$= 0.22128$
$k_4 = 0.1 (1 + .1,\ 2 + .22128)$
$= 0.244164$
$k = 1/6(k_1 + 2k_2 + 2k_3 + k_4) = 1.32824/6 = 0.22137$
$y_1 = k + y_0 = 2 + 1.32824/6$
$= 2.2213$

For finding y_2:- $k_1 = 0.1 f(x_1, y_1) = 0.244165$
$k_2 = 0.1 f(1.1 + .1/2,\ 2.2213 + \frac{.244165}{2})$
$= 0.26926$
$k_3 = 0.1 f(1.1 + .1/2,\ 2.2213 + .26926/2)$
$= 0.269884$
$k_4 = 0.1 (1.1 + .1,\ 2.2213 + .269884)$
$= 0.29736$
$k = 1/6(k_1 + 2k_2 + 2k_3 + k_4) = 0.26997$
$y_2 = y_1 + k = 2.4913$

Solution 7 (b)

Solution :-

$$f(x) = x^2 - 5x + 2 = 0$$

$$f(0) = 2 > 0$$

$$f(1) = -2 < 0$$

and let $x_0 = 0.6$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - 5x_n + 2)}{2x_n - 5}$$

$$= \frac{2x_n^2 - \cancel{5x_n} - x_n^2 + \cancel{5x_n} - 2}{2x_n - 5}$$

$$= \frac{x_n^2 - 2}{2x_n - 5}$$

$x_1 = -0.2158$

$x_2 = 0.3596$

$x_3 = 0.4370$

$x_4 = 0.4384$

$x_5 = 0.4384$

Hence the real root of the equation $x^2 - 5x + 2 = 0$ is 0.4384 Ans

Govt. of Bihar
MUZAFFARPUR INSTITUTE OF TECHNOLOGY, MUZAFFARPUR, BIHAR – 842003

(Under the department of Science & Technology, Bihar, Patna)

B. Tech 2nd Semester Mid-Term Examination, 2019

Mathematics-II

(CE)

Time: 2 hours **Full Marks: 20**

Subject Code: 211202

Attempt any four questions out of which question no. 1 is compulsory.

1. Chose the correct option of the following: (1x5=5 Marks)

(a) Ans: (iii) Δ

(b) Ans: (i) $x_{n+1} = x_n - \dfrac{f(x_n)}{f^{/}(x_n)}$

(c) Ans: (ii) $|x| < 1$

(d) Ans: (iv) -1

(e) Ans: (iii) $2xy + C$

2. (a) Define Analytic function and State the Necessary and Sufficient Condition for Analytic Function.

Solution: A function f(z) is said to be analytic at a point z=a if f(z) is differentiable not only at z=a but differentiable at each point in some nbd of z=a.

A function is analytic in a domain if it is analytic at each point of the domain.

Ex. (i) $f(z) = e^z$ is analytic everywhere.

(ii) $f(z) = |z|^2$ is differentiable at z=0 but not analytic at z=0. Reason is that f (z) is differentiable at z=0 only.

Necessary and sufficient conditions for a function to be analytic

The necessary and sufficient conditions for a function $f(z) = u(x, y) + iv(x, y)$, to be analytic are that:

1. The four partial derivatives of its real and imaginary parts $\partial u\ \partial x$, $\partial v\ \partial y$, $\partial u\ \partial y$, $\partial v\ \partial x$ are continuous.
2. The four partial derivatives of its real and imaginary parts $\partial u\ \partial x$, $\partial v\ \partial y$, $\partial u\ \partial y$, $\partial v\ \partial x$ satisfy the Cauchy-Riemann equations.

   i.e. $u_x = v_y$ and $u_y = -v_x$.

(b) If $u-v=(x-y)(x^2+4xy+y^2)$ and $f(z)=u+iv,$ is an analytic function of $z=x+iy,$ find $f(z)$ in terms of z.

Solution:- Let $f(z)=u(x,y)+iv(x,y)$ be analytic function

Given: $u-v=(x-y)(x^2+4xy+y^2)$ —— (i)

Differentiating (i) partially w.r.t. x; we get

$u_x-v_x=(x-y)(2x+4y)+(x^2+4xy+y^2)$

$\Rightarrow$ $u_x+u_y=3x^2+6xy-3y^2$ —— (ii) $\left[\because \begin{array}{c}\text{C-R eqn.}\\ v_x=-u_y\end{array}\right]$

Again differentiating (i) partially w.r.t. y; we get

$u_y-v_y=(x-y)(4x+2y)-(x^2+4xy+y^2)$

$\Rightarrow$ $u_y-u_x=3x^2-6xy-3y^2$ —— (iii) $\left[\because \begin{array}{c}\text{CR eqn}\\ u_x=v_y\end{array}\right]$

Now (ii) + (iii) $\Rightarrow$ $2u_y=6x^2-6y^2$

$\Rightarrow$ $u_y=3x^2-3y^2$ —— (iv)

(ii) $\Rightarrow$ $u_x=6xy$

Now By Milne's Thomson Theorem;

$f(z)=\int (u_x)_{\substack{x=z\\y=0}}\,dz-i\int (u_y)_{\substack{x=z\\y=0}}\,dz+c$

$f(z)=\int 0\,dz-i\int 3z^2\,dz+c$

$\therefore$ $\boxed{f(z)=-iz^3+c}$ is the required analytic fn.

3. (a) Show that the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin even though C-R equations are satisfied thereof.

Sol^n:- Let $f(z)=u(x,y)+iv(x,y)=\sqrt{|xy|}$

So that $u(x,y)=\sqrt{|xy|}$ and $v(x,y)=0$

We have, at the origin

$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0$

$\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$

$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{0-0}{x}=0,\quad \frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{0-0}{y}=0$

Hence the Cauchy-Riemann equations are satisfied at the origin

Now $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{z\to 0}\frac{\sqrt{|xy|}-0}{x+iy}$

Letting $z\to 0$ along $y=mx$, we get

$f'(0)=\lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x(1+im)}=\frac{\sqrt{|m|}}{(1+im)}$

This limit is not unique since it depends on m.

Hence $f'(0)$ does not exist and so $f(z)$ is not analytic at $z=0$.

(b) Determine an Analytic function $f(z)$ in terms of z whose real part is $e^x(x\sin y - y\cos y)$.

Given $u = e^{-x}(y\cos y - x\sin y)$

$$\frac{\partial u}{\partial x} = e^{-x}(-\sin y) - e^{-x}(y\cos y - x\sin y)$$

$$\frac{\partial u}{\partial x} = -e^{-x}(\sin y + y\cos y - x\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(\sin y + y\cos y - x\sin y) - e^{-x}(-\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(2\sin y + y\cos y - x\sin y)$$

Hly: $\frac{\partial u}{\partial y} = e^{-x}(\cos y - y\sin y - x\cos y)$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-\sin y - y\cos y - \sin y + x\sin y)$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

and hence $u(x,y) = e^{-x}(y\cos y - x\sin y)$ is harmonic function.

Now for conjugate harmonic $v(x,y)$.

By differential eqn method, we have;

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy \quad —(1)$$

[Since $f(z) = u + iv$ is analytic $\Rightarrow$ $f(z)$ satisfies C-R eqn $\Rightarrow$ $u_x = v_y$ and $v_x = -u_y$]

Since R.H.S of (1) is exact differential eqn.

Therefore, the solution of (1) is given by

$$\int dv = \underbrace{\int\left(-\frac{\partial u}{\partial y}\right)dx}_{\text{treating } y \text{ as constant}} + \underbrace{\int\left(\frac{\partial u}{\partial x}\right)dx}_{\text{only those terms independent of } x} + c$$

$$\Rightarrow \int dv = \int -e^{-x}(\cos y - y\sin y - x\cos y)\,dx + \int 0\,dx + c$$

$$= -\int e^{-x}\cos y\,dx + y\sin y\int e^{-x}dx + \cos y\int xe^{-x}dx + c$$

$$= +\cos y\,e^{-x} - y\sin y\,e^{-x} + \cos y\left[-xe^{-x} - e^{-x}\right] + c$$

$$= e^{-x}(\cos y - y\sin y - x\cos y - \cos y) + c$$

$$\boxed{v = -e^{-x}(y\sin y + x\cos y) + c}$$

and hence

$$f(z) = u(x,y) + iv(x,y)$$

$$= e^{-x}(y\cos y - x\sin y) + i\left[-e^{-x}(y\sin y + x\cos y) + c\right]$$

$$= e^{-x}(y\cos y - x\sin y) - ie^{-x}(y\sin y + x\cos y) + k$$

where $k = ic$

4. (a) Solve $xe^{x}\left(dx-dy\right)+e^{x}\,dx+ye^{y}dy=0$.

Solution

Given eqn :

$$xe^{x}(dx-dy)+e^{x}dx+ye^{y}dy=0$$

$$\Rightarrow \quad dx\left(xe^{x}+e^{x}\right)+dy\left(ye^{y}-xe^{x}\right)=0 \quad \text{--- (1)}$$

eqn (1) is in the form of $M\,dx+N\,dy=0$ — (2)

where $M = xe^{x}+e^{x}$

$N = ye^{y}-xe^{x}$

$$\frac{\partial M}{\partial y}=0, \quad \frac{\partial N}{\partial x} = -xe^{x}-e^{x} = -e^{x}(x+1)$$

$\because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ eqn (1) is not exact

Now $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) = \frac{1}{(x+1)e^{x}}\left[-e^{x}(x+1)-0\right]$

$= -1$

$$I.F = e^{\int(-1)dy} = e^{-y} = \frac{1}{e^{y}}$$

~~eqn (1) reduces~~ $\left(\frac{xe^{x}+e^{x}}{e^{y}}\right)dx+\left(\frac{ye^{y}-xe^{x}}{e^{y}}\right)dy=0$.

(1) $\Rightarrow$ — (3)

eqn (3) is exact

Therefore the req. soln is given by

$$\int\left(\frac{xe^{x}+e^{x}}{e^{y}}\right)dy + \int\frac{ye^{y}}{e^{y}}dy = c$$

treating y as constant — term independent of x.

(b) Solve $\left(D^2-2D+1\right)y = xe^x \sin x$.

Solution:

C.F.: Here $m^2 - 2m + 1 = 0$ is the A.E. with $m = 1$ as a double root so that the C.F. y_c is

$$y_c = (c_1 + c_2 x)e^x$$

$$\text{P.I.: } y_p = \frac{1}{(D^2 - 2D + 1)} x(e^x \sin x)$$

$$= \frac{1}{(D-1)^2} e^x (x \sin x)$$

using shift result with $a = 1$ so that D is replaced by $D + 1$, we get

$$y_p = \frac{e^x}{[(D+1)-1]^2}(x \sin x) = \frac{e^x}{D^2} x \sin x$$

Applying result VI

$$\frac{1}{D^2} x(\sin x) = x \cdot \frac{1}{D^2} \sin x - \frac{2D}{D^4} \sin x \tag{5}$$

$$= x(-\sin x) - 2\cos x \tag{6}$$

Thus $\quad y_p = e^x[-x \sin x - 2\cos x] \quad (7)$

Hence G.S.: $y = y_c + y_p$

$$y = (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x)$$

5. (a) Solve $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$.

Solution: Substituting (2), (4), (5), (6) in the given D.E., we get

$$\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y + 3\mathcal{D}(\mathcal{D}-1)y + \mathcal{D}y + y = e^t + t$$

or

$$\mathcal{D}^3 y + y = e^t + t$$

C.F.: The A.E. is $m^3 + 1 = 0$ having roots $m = -1, \frac{1 \pm \sqrt{3}i}{2}$ so that the C.F. y_c is

$$y_c = c_1 e^{-t} + e^{\frac{t}{2}} \left(c_2 \cos \frac{\sqrt{3}}{2} t + c_3 \sin \frac{\sqrt{3}}{2} t \right)$$

P.I. : $y_p = \frac{1}{\mathcal{D}^3 + 1}\{e^t + t\} = \frac{1}{\mathcal{D}^3+1} e^t + \frac{1}{\mathcal{D}^3+1} t$

$$= \frac{1}{1^3+1} e^t + \{1 - \mathcal{D}^3 + \mathcal{D}^6 + \ldots\} t$$

$$= \frac{e^t}{2} + t - 0 + 0 + \ldots$$

(b) Solve $\left(D^2+2D+1\right)y=e^{-x}\log x$ by using method of variation of parameters.

Solution:

C.F.: Here A.E. is $m^2+2m+1=0$ with $m=-1$ as the double root so that the C.F. y_c is

$$y_c=(c_1+c_2x)e^{-x}.$$

Take $y_1=e^{-x}$ and $y^2=xe^{-x}$ as the fundamental system. Now the Wronskian w is

$$w=y_1y_2'-y_2y_1'=e^{-x}(e^{-x}-xe^{-x})$$
$$-(xe^{-x})(-e^{-x})=e^{-2x}$$

Assume the P.I. y_p as

$$y_p=u_1(x)y_1(x)+u_2(x)y_2(x)$$

where $u_1=-\int\frac{fy_2}{w}dx$ with $f=e^{-x}\cdot\ln x$

$$u_1=-\int\frac{e^{-x}\cdot\ln x\cdot xe^{-x}}{e^{-2x}}dx$$
$$=-\int x\cdot\ln x\cdot dx$$
$$u_1=\frac{x^2}{2}\left[\frac{1}{2}-\ln x\right]$$

Also

$$u_2=\int\frac{fy_2}{w}dx=\int\frac{e^{-x}\ln x\cdot e^{-x}}{e^{-2x}}dx$$
$$=\int\ln x\,dx=x\ln x-x$$

Thus $y_p=\frac{x^2}{2}\left[\frac{1}{2}-\ln x\right]e^{-x}+e^{-2x}[x\ln x-x]$.
Hence G.S: $y=y_c+y_p$

$$y=c_1e^{-x}+c_2xe^{-x}+\frac{x^2}{2}\left(\frac{1}{2}-\ln x\right)e^{-x}$$
$$+e^{-2x}(x\cdot\ln x-x)$$

6. (a) Using Runge-Kutta method, Solve the equation

$$\frac{dy}{dx} = x^3 + \frac{y}{2} \ , \ y(1) = 2 \quad \text{for} \quad y(1.1) \quad \text{and} \quad y(1.2).$$

Sol^n:- Here $f(x,y) = x^3 + y/2$

and $h = 0.1$, $x_0 = 1$, $y_0 = 2$, $x_1 = 1.1$, $x_2 = 1.2$

For finding y_1:-

$K_1 = hf(x_0, y_0) = 0.1 f(1, 2)$

$= 0.2$

$K_2 = 0.1 f(1 + .1/2, 2 + .2/2)$

$= 0.22076$

$K_3 = 0.1 f(1 + .1/2, .22076/2 + 2)$

$= 0.22128$

$K_4 = 0.1 (1 + .1, 2 + .22128)$

$= 0.244164$

$K = 1/6 (K_1 + 2K_2 + 2K_3 + K_4) = 1.32824/6 = 0.22137$

$y_1 = K + y_0 = 2 + 1.32824/6$

$= 2.2213$

For finding y_2:- $K_1 = 0.1 f(x_1, y_1) = 0.244165$

$K_2 = 0.1 f(1.1 + .1/2, 2.2213 + \frac{.244165}{2})$

$= 0.26926$

$K_3 = 0.1 f(1.1 + .1/2, 2.2213 + .26926/2)$

$= 0.269884$

$K_4 = 0.1 (1.1 + .1, 2.2213 + .269884)$

$= 0.29736$

$K = 1/6 (K_1 + 2K_2 + 2K_3 + K_4) = 0.26997$

$y_2 = y_1 + K = 2.4913$

(b) Find a real root of the equation $x^2 - 5x + 2 = 0$ by Newton-Raphson's method.

Solution:-

$f(x) = x^2 - 5x + 2 = 0$

$f(0) = 2 > 0$

$f(1) = -2 < 0$

and let $x_0 = 0.6$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - 5x_n + 2)}{2x_n - 5}$$

$$= \frac{2x_n^2 - \cancel{5x_n} - x_n^2 + \cancel{5x_n} - 2}{2x_n - 5}$$

$$= \frac{x_n^2 - 2}{2x_n - 5}$$

$x_1 = -0.2158$

$x_2 = 0.3596$

$x_3 = 0.4370$

$x_4 = 0.4384$

$x_5 = 0.4384$

Hence the real root of the equation $x^2 - 5x + 2 = 0$ is 0.4384 Ans

7. (a) Find the value of $J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Sol^n. From Recurrence Relation; we have

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad \text{——} (1)$$

put $n = 1, 2, 3$ in equation (1); we get

for $n=1$; $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

for $n=2$; $J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$

for $n=3$; $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$

$$= \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \frac{24}{x^2}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - \frac{6}{x} J_1(x) - \frac{2}{x} J_1(x) + J_0$$

$$= \frac{48}{x^3} J_1(x) - \frac{24}{x^2} J_0(x) - \frac{6}{x} J_1(x) - \frac{2}{x} J_1(x) + J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$$

(b) Prove that $\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$.

Proof: As we have the solution of Bessel's equation is as

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} \lfloor m \, \Gamma(m+n+1)}$$

So; $$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{d}{dx}\left[\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+2n}}{2^{2m+n} \lfloor m \, \Gamma(m+n+1)}\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2n) x^{2m+2n-1}}{2^{2m+n} \lfloor m \, (m+n) \Gamma(m+n)}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m 2 \cancel{(m+n)} x^{2m+(n-1)}}{2^{2m+n} \lfloor m \, \cancel{(m+n)} \Gamma(m+n)}$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+(n-1)}}{2^{2m+(n-1)} \lfloor m \, \Gamma[m+(n-1)+1]}$$

$$\boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}$$

Proved

Govt. of Bihar
MUZAFFARPUR INSTITUTE OF TECHNOLOGY, MUZAFFARPUR, BIHAR – 842003

(Under the department of Science & Technology, Bihar, Patna)

B. Tech 2nd Semester Mid-Term Examination, 2019

Mathematics-II

(ME)

Time: 2 hours **Full Marks: 20**

Subject Code: 211202

Attempt any four questions out of which question no. 1 is compulsory.

1. Chose the correct option of the following: (1x5=5 Marks)

(a) Ans: (iii) Δ

(b) Ans: (i) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

(c) Ans: (ii) $|x| < 1$

(d) Ans: (iv) -1

(e) Ans: (iii) $2xy + C$

2. (a) Define Analytic function and State the Necessary and Sufficient Condition for Analytic Function.

Solution: A function f(z) is said to be analytic at a point z=a if f(z) is differentiable not only at z=a but differentiable at each point in some nbd of z=a.

A function is analytic in a domain if it is analytic at each point of the domain.

Ex. (i) $f(z) = e^z$ is analytic everywhere.

(ii) $f(z) = |z|^2$ is differentiable at z=0 but not analytic at z=0. Reason is that f (z) is differentiable at z=0 only.

Necessary and sufficient conditions for a function to be analytic

The necessary and sufficient conditions for a function $f(z) = u(x, y) + iv(x, y)$, to be analytic are that:

1. The four partial derivatives of its real and imaginary parts $\partial u\ \partial x$, $\partial v\ \partial y$, $\partial u\ \partial y$, $\partial v\ \partial x$ are continuous.
2. The four partial derivatives of its real and imaginary parts $\partial u\ \partial x$, $\partial v\ \partial y$, $\partial u\ \partial y$, $\partial v\ \partial x$ satisfy the Cauchy-Riemann equations.

   i.e. $u_x = v_y$ and $u_y = -v_x$.

(b) If $u - v = (x-y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv,$ is an analytic function of $z = x + iy,$ find $f(z)$ in terms of z.

Solution:- Let $f(z) = u(x,y) + i\,v(x,y)$ be analytic function

Given: $u - v = (x-y)(x^2+4xy+y^2)$ —— (i)

Differentiating (i) partially w.r.t x; we get

$u_x - v_x = (x-y)(2x+4y) + (x^2+4xy+y^2)$

$\Rightarrow u_x + u_y = 3x^2 + 6xy - 3y^2$ —— (ii) $\left[\because \begin{array}{c}\text{C-R eqn}\\ v_x = -u_y\end{array}\right]$

Again differentiating (i) partially w.r.t y; we get

$u_y - v_y = (x-y)(4x+2y) - (x^2+4xy+y^2)$

$\Rightarrow u_y - u_x = 3x^2 - 6xy - 3y^2$ —— (iii) $\left[\because \begin{array}{c}\text{CR eqn}\\ u_x = v_y\end{array}\right]$

Now (i) + (ii) $\Rightarrow 2u_y = 6x^2 - 6y^2$

$\Rightarrow u_y = 3x^2 - 3y^2$ —— (iv)

(ii) $\Rightarrow u_x = 6xy$

Now By Milne's Thomson Theorem;

$f(z) = \int (u_x)_{\substack{x=z\\y=0}} dz - i \int (u_y)_{\substack{x=z\\y=0}} dz + c$

$f(z) = \int 0\, dz - i \int 3z^2\, dz + c$

$\therefore \boxed{f(z) = -iz^3 + c}$ is the required analytic fn.

3. (a) Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin even though C-R equations are satisfied thereof.

Sol^n:- Let $f(z) = u(x,y) + i\,v(x,y) = \sqrt{|xy|}$

So that $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$

We have, at the origin

$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{0-0}{x} = 0, \quad \frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{0-0}{y} = 0$

Hence the Cauchy-Riemann equations are satisfied at the origin

Now $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{z\to 0} \frac{\sqrt{|xy|} - 0}{x+iy}$

Letting $z \to 0$ along $y = mx$, we get

$f'(0) = \lim_{x\to 0} \frac{\sqrt{|mx^2|}}{x(1+im)} = \frac{\sqrt{|m|}}{(1+im)}$

This limit is not unique since it depends on m.

Hence $f'(0)$ does not exist and so $f(z)$ is not analytic at $z=0$.

(b) Determine an Analytic function $f(z)$ in terms of z whose real part is $e^{x}(x\sin y - y\cos y)$.

)(b) Given $u = e^{-x}(y\cos y - x\sin y)$

$$\frac{\partial u}{\partial x} = e^{-x}(-\sin y) - e^{-x}(y\cos y - x\sin y)$$

$$\frac{\partial u}{\partial x} = -e^{-x}(\sin y + y\cos y - x\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(\sin y + y\cos y - x\sin y) - e^{-x}(-\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(2\sin y + y\cos y - x\sin y)$$

Lly: $\frac{\partial u}{\partial y} = e^{-x}(\cos y - y\sin y - x\cos y)$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-\sin y - y\cos y - \sin y + x\sin y)$$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

and hence $u(x, y) = e^{-x}(y\cos y - x\sin y)$ is harmonic function.

Now for conjugate harmonic $v(x, y)$.

By differential eqn method, we have;

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy \quad -(1)$$

[Since $f(z) = u + iv$ is analytic $\Rightarrow f(z)$ satisfies C-R eqn $\Rightarrow u_x = v_y$ and $v_x = -u_y$

Since R.H.S of (1) is exact differential eqn.

Therefore, the solution of (1) is given by

$$\int dv = \underbrace{\int\left(-\frac{\partial u}{\partial y}\right)dx}_{\text{treating } y \text{ as constant}} + \underbrace{\int\left(\frac{\partial u}{\partial x}\right)dx}_{\text{only those terms independent of } x} + c$$

$$\Rightarrow \int dv = \int -e^{-x}(\cos y - y\sin y - x\cos y)\,dx$$

$$+ \int 0\,dx + c$$

$$= -\int e^{-x}\cos y\,dx + y\sin y\int e^{-x}dx + \cos y\int xe^{-x}dx + c$$

$$= +\cos y\,e^{-x} - y\sin y\,e^{-x} + \cos y\left[-xe^{-x} - e^{-x}\right] + c$$

$$= e^{-x}(\cos y - y\sin y - x\cos y - \cos y) + c$$

$$\boxed{v = -e^{-x}(y\sin y + x\cos y) + c}$$

and hence

$$f(z) = u(x, y) + iv(x, y)$$

$$= e^{-x}(y\cos y - x\sin y) + i\left[-e^{-x}(y\sin y + x\cos y) + c\right]$$

$$= e^{-x}(y\cos y - x\sin y) - ie^{-x}(y\sin y + x\cos y) + k$$

where $k = ic$

4. (a) Solve $xe^{x}\left(dx-dy\right)+e^{x}\,dx+ye^{y}dy=0$.

Solution

Given eqn :

$x e^{x}(dx-dy) + e^{x}dx + ye^{y}dy = 0$

$\Rightarrow$ $dx\,(xe^{x}+e^{x}) + dy\,(ye^{y}-xe^{x}) = 0$ —(1)

eqn (1) is in the form of $M\,dx + N\,dy = 0$ —(2)

where $M = xe^{x}+e^{x}$

$N = ye^{y} - xe^{x}$

$\dfrac{\partial M}{\partial y} = 0$, $\dfrac{\partial N}{\partial x} = -xe^{x} - e^{x} = -e^{x}(x+1)$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ eqn (1) is not exact

Now $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right) = \dfrac{1}{(x+1)e^{x}}\left[-e^{x}(x+1)-0\right]$

$= -1$

$I.F = e^{\int(-1)dy} = e^{-y} = \dfrac{1}{e^{y}}$

~~eqn (1) reduces~~ $\left(\dfrac{xe^{x}+e^{x}}{e^{y}}\right)dx + \left(\dfrac{ye^{y}-xe^{x}}{e^{y}}\right)dy = 0$

(1) $\Rightarrow$ —(3)

eqn (3) is exact

Therefore the req. soln is given by

$\int\left(\dfrac{xe^{x}+e^{x}}{e^{y}}\right)dy + \int\dfrac{ye^{y}}{e^{y}}dy = c$

treating y as constant — term independent of x.

(b) Solve $\left(D^2-2D+1\right)y = xe^x \sin x$.

Solution:

C.F.: Here $m^2 - 2m + 1 = 0$ is the A.E. with $m = 1$ as a double root so that the C.F. y_c is

$$y_c = (c_1 + c_2 x)e^x$$

P.I.: $y_p = \dfrac{1}{(D^2 - 2D + 1)} x(e^x \sin x)$

$$= \frac{1}{(D-1)^2} e^x (x \sin x)$$

using shift result with $a = 1$ so that D is replaced by $D + 1$, we get

$$y_p = \frac{e^x}{[(D+1)-1]^2}(x \sin x) = \frac{e^x}{D^2} x \sin x$$

Applying result VI

$$\frac{1}{D^2} x(\sin x) = x \cdot \frac{1}{D^2} \sin x - \frac{2D}{D^4} \sin x \tag{5}$$

$$= x(-\sin x) - 2\cos x \tag{6}$$

Thus $\quad y_p = e^x[-x \sin x - 2\cos x] \quad (7)$

Hence G.S.: $y = y_c + y_p$

$$y = (c_1 + c_2 x)e^x - e^x(x \sin x + 2\cos x)$$

5. (a) Solve $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x + \log x$.

Solution: Substituting (2), (4), (5), (6) in the given D.E., we get

$$\mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2)y + 3\mathcal{D}(\mathcal{D}-1)y + \mathcal{D}y + y = e^t + t$$

or

$$\mathcal{D}^3 y + y = e^t + t$$

C.F.: The A.E. is $m^3 + 1 = 0$ having roots $m = -1, \frac{1 \pm \sqrt{3}i}{2}$ so that the C.F. y_c is

$$y_c = c_1 e^{-t} + e^{\frac{t}{2}} \left(c_2 \cos \frac{\sqrt{3}}{2} t + c_3 \sin \frac{\sqrt{3}}{2} t \right)$$

P.I. : $y_p = \dfrac{1}{\mathcal{D}^3 + 1}\{e^t + t\} = \dfrac{1}{\mathcal{D}^3 + 1} e^t + \dfrac{1}{\mathcal{D}^3 + 1} t$

$$= \frac{1}{1^3 + 1} e^t + \{1 - \mathcal{D}^3 + \mathcal{D}^6 + \ldots\} t$$

$$= \frac{e^t}{2} + t - 0 + 0 + \ldots$$

(b) Solve $\left(D^2+2D+1\right)y = e^{-x}\log x$ by using method of variation of parameters.

Solution:

C.F.: Here A.E. is $m^2+2m+1=0$ with $m=-1$ as the double root so that the C.F. y_c is

$$y_c = (c_1 + c_2 x)e^{-x}.$$

Take $y_1 = e^{-x}$ and $y^2 = xe^{-x}$ as the fundamental system. Now the Wronskian w is

$$w = y_1 y_2^i - y_2 y_1' = e^{-x}(e^{-x} - xe^{-x})$$
$$-(xe^{-x})(-e^{-x}) = e^{-2x}$$

Assume the P.I. y_p as

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where $u_1 = -\int \frac{fy_2}{w}dx$ with $f = e^{-x}\cdot \ln x$

$$u_1 = -\int \frac{e^{-x}\cdot \ln x \cdot xe^{-x}}{e^{-2x}}dx$$
$$= -\int x\cdot \ln x \cdot dx$$
$$u_1 = \frac{x^2}{2}\left[\frac{1}{2} - \ln x\right]$$

Also

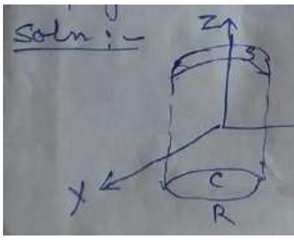
$$u_2 = \int \frac{fy_2}{w}dx = \int \frac{e^{-x}\ln x \cdot e^{-x}}{e^{-2x}}dx$$
$$= \int \ln x dx = x\ln x - x$$

Thus $y_p = \frac{x^2}{2}\left[\frac{1}{2} - \ln x\right]e^{-x} + e^{-2x}[x\ln x - x]$.
Hence G.S: $y = y_c + y_p$

$$y = c_1 e^{-x} + c_2 x e^{-x} + \frac{x^2}{2}\left(\frac{1}{2} - \ln x\right)e^{-x}$$
$$+e^{-2x}(x\cdot \ln x - x)$$

6. (a) Verify Stoke's theorem for the vector field $\vec{F} = (2x - y)\hat{i} - yz^2 j - y^2 zk$ over the upper half surface of sphere $x^2 + y^2 + z^2 = 1$, bounded by its projection on xy- plane.

Soln :-



Let S be upper half surface of Sphere $x^2+y^2+z^2=1$. Its projection on xy plane is circle C of radius unity and centre O

The eqn of C is $x^2+y^2=1,\ z=0$

$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_C [(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}]\cdot[\hat{i}dx+\hat{j}dy+\hat{k}dz]$

$= \int_C [(2x-y)dx - yz^2 dy - y^2 z\, dz]$

$= \int_C (2x-y)dx \quad [\because \text{on } C,\ z=0,\ dz=0]$

The parametric representation of circle $x^2+y^2=1$ is

$x=\cos\theta,\ y=\sin\theta,\ z=0,\quad 0\le\theta\le 2\pi$

$\therefore \int \vec{F}\cdot d\vec{r} = \int_C (2x-y)dx = \int_0^{2\pi}(2\cos\theta - \sin\theta)\frac{dx}{d\theta}d\theta$

$= \int_0^{2\pi}(2\cos\theta-\sin\theta)(-\sin\theta)d\theta = \int_0^{2\pi}(-\sin 2\theta + \sin^2\theta)d\theta$

$= \int_0^{2\pi}\left(-\sin 2\theta + \frac{1-\cos 2\theta}{2}\right)d\theta = \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$= \frac{1}{2} + \pi - \frac{1}{2} = \pi$

Also $\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{k}$

$\text{Curl}\,\vec{F}\cdot\hat{n} = \hat{k}\cdot\hat{n} = \hat{n}\cdot\hat{k}$

and $\iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S \hat{n}\cdot\hat{k}\,ds = \iint_S \hat{n}\cdot\hat{k}\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$= \iint_R dx\,dy$ where R is projection of S on xy-plane.

$= \int_{x=-1}^{1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy = \int_{-1}^{1} 2\sqrt{1-x^2}\,dx = 4\int_0^1 \sqrt{1-x^2}\,dx$

$= 4\left|\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1}x\right|_0^1 = 4\left[\frac{1}{2}\cdot\frac{\pi}{2}\right] = \pi$

Since $\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds = \pi$

Hence Stoke's theorem is verified.

(b). Using divergence theorem, evaluate $\int_s \vec{r}.n\,ds$ where s is the surface of the sphere $x^2+y^2+z^2=9$.

Soln:— By Gauss divergence theorem, we have

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{r}\,dv = \iiint_V \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})\,dv$$

$$= \iiint_V 3\,dv = 3V \quad \left\{\because \iiint_V dv = \text{volume of closed surface of sphere of radius } 3\right.$$

$$= 3\times\frac{4}{3}\pi\times 3^3$$

$$= 108\pi \quad \underline{\text{Ans}}$$

7. (a) Using Runge-Kutta method, Solve the equation

$$\frac{dy}{dx}=x^3+\frac{y}{2}\ ,\quad y(1)=2 \quad \text{for} \quad y(1.1) \quad \text{and} \quad y(1.2).$$

Solⁿ:- Here $f(x,y) = x^3 + y/2$

and $h = 0.1$, $x_0 = 1$, $y_0 = 2$, $x_1 = 1.1$, $x_2 = 1.2$

For finding y_1:-

$K_1 = hf(x_0, y_0) = 0.1 f(1, 2)$

$= 0.2$

$K_2 = 0.1 f(1 + .1/2,\ 2 + .2/2)$

$= 0.22076$

$K_3 = 0.1 f(1 + .1/2,\ .22076/2 + 2)$

$= 0.22128$

$K_4 = 0.1 (1 + .1,\ 2 + .22128)$

$= 0.244164$

$K = 1/6(K_1 + 2K_2 + 2K_3 + K_4) = 1.32824/6 = 0.22137$

$y_1 = K + y_0 = 2 + 1.32824/6$

$= 2.2213$

For finding y_2:- $K_1 = 0.1 f(x_1, y_1) = 0.244165$

$K_2 = 0.1\ f(1.1 + .1/2,\ 2.2213 + \frac{.244165}{2})$

$= 0.26926$

$K_3 = 0.1 f(1.1 + .1/2,\ 2.2213 + .26926/2)$

$= 0.269884$

$K_4 = 0.1 (1.1 + .1,\ 2.2213 + .269884)$

$= 0.29736$

$K = 1/6(K_1 + 2K_2 + 2K_3 + K_4) = 0.26997$

$y_2 = y_1 + K = 2.4913$

(b) Find a real root of the equation $x^2 - 5x + 2 = 0$ by Newton-Raphson's method.

Solution :-

$$f(x) = x^2 - 5x + 2 = 0$$

$$f(0) = 2 > 0$$

$$f(1) = -2 < 0$$

and let $x_0 = 0.6$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - 5x_n + 2)}{2x_n - 5}$$

$$= \frac{2x_n^2 - \cancel{5x_n} - x_n^2 + \cancel{5x_n} - 2}{2x_n - 5}$$

$$= \frac{x_n^2 - 2}{2x_n - 5}$$

$x_1 = -0.2158$

$x_2 = 0.3596$

$x_3 = 0.4370$

$x_4 = 0.4384$

$x_5 = 0.4384$

Hence the real root of the equation $x^2 - 5x + 2 = 0$ is 0.4384 Ans